Integrated Fault/State Estimation for Two-Dimensional Linear Time-Varying Systems

1st Liang Cao
College of Information Science and Technology
Beijing University of Chemical Technology
Beijing, China
13269226796@163.com

2nd Dong Zhao
College of Information Science and Technology
Beijing University of Chemical Technology
Beijing, China
autozhaodong@gmail.com

3th Youqing Wang*
College of Electrical Engineering and Automation
Shandong University of Science and Technology
Qingdao, China
wang.youqing@ieee.org

***Abstract*—In this study, state and fault estimation problem are addressed for two-dimensional linear time-varying systems, which are described by Fornasini-Marchesini second model. For the systems subject to fault and white noise, based on generalized two-dimensional Kalman estimation theory, the system state and fault are estimated simultaneously in the sense of unbiased minimum variance. By transforming the existence condition of estimator into the solvability of matrix equations, the desired estimator gain matrices can be deduced. A numerical example is given to demonstrate the effectiveness of the proposed method in the end.**

Keywords—fault/state estimation, two dimensional systems, time-varying systems, unbiased minimum variance.

I. Introduction

Systems that update from two independent dimensions are called two-dimensional (2-D) systems [1-7]. 2-D systems are wildly existed in many engineering fields and can be used to solve many complex problems, such as the achievement of multi-variable network [8], multidimensional digital filtering and signal processing [9], image data processing [2] and batch process control [10]. In practice, 2-D systems face tremendous threat of fault and have the needs of reliability and security, if fault occurs in sensor or actuator, the system performance will be degraded. However, due to the complexity of 2-D systems, the fault diagnosis and fault-tolerant control researches about 2-D systems are rare, leaving many open problems to be settled. Hence, we focus on the fault estimation of 2-D systems in this paper to fulfil the application requirement.

Observer/filter-based methods play significant role in model based fault diagnosis and they can provide information such as fault sizes and kinds. Therefore, there are important industrial and theoretical applications in fault diagnosis [11-14] and fault tolerant control [15, 16]. However, most of these studies are focused on 1-D systems. Note that, some studies about 2-D observers/filters have just appeared in the literature. For example, in [17], 2-D filter is used to obtain the fault detection results. In [18], based on 2-D generalized KYP lemma, the design method of 2-D fault detection filter is given by linear matrix inequality technique. As is well known, fault diagnosis has a close relationship with fault estimation and state estimation. For example, based on 2-D singular system theory, the state and the sensor fault are estimated simultaneously in [3]. For 2-D systems with time-varying delays and system perturbations, the state and fault estimation are obtained by using 2-D robust control theory in [19]. Although there are many results about 2-D systems estimation, compared with one dimensional (1-D) systems, the estimation of 2-D systems is far from being well-investigated, especially in the problem of simultaneous estimation of state and fault.

Time-varying systems [20, 21] are widely existed in engineering practice, and most real-time systems have time-varying system parameters [22]. However, due to the complexity of mathematics and computation, comparing with the rich results of time-invariant systems, the corresponding studies on time-varying systems are still in initial stage and far from satisfactory. There are a few preliminary results on the estimation of 2-D time-varying systems. In [23], the minimum variance state estimator is obtained by using 2-D generalized Kalman filtering theory. Based on projection, the solution of 2-D Kalman filter for 2-D time-varying systems is given in [24]. However, the effects of faults in state and measurement are not considered in these results. Moreover, the calculation process of their estimators is very complicated.

Motivated by aforementioned discussion, this paper considers fault in state and measurement equation. In the sense of unbiased minimum variance, for 2-D time-varying systems with disturbance, we firstly design a new observer that can simultaneously estimate state and fault. Furthermore, the necessary and sufficient conditions for the existence of unbiased estimator and unbiased minimum variance estimator are derived.

Notation. Throughout this paper, $X \in R^{m\times n}$ means that X is an $m\times n$ real matrix; I and $\mathbf{0}$ indicate an identity matrix and zero matrix, respectively; X^{+} is the Moore-Penrose pseudo-inverse of matrix X ; $\mathbb{E}[\bullet]$ is the mathematical expectation of variable of $\bullet$.

978-1-5386-3524-7/17/$31.00 ©2017 IEEE

II. Problem Formulation

The 2-D time-varying systems with disturbances and faults are given as follows:

$$\begin{cases} x(i+1,j+1)=A_1(i,j+1)x(i,j+1) \\ +A_2(i+1,j)x(i+1,j) \\ +D_1(i,j+1)d(i,j+1)+D_2(i+1,j)d(i+1,j) \\ +F_1(i,j+1)f(i,j+1)+F_2(i+1,j)f(i+1,j) \\ y(i,j)=C(i,j)x(i,j)+D_3(i,j)v(i,j) \\ +F_3(i,j)f(i,j) \end{cases} \tag{1}$$

where $x(i,j)\in R^n$, $y(i,j)\in R^p$, $u(i,j)\in R^m$ are the system state vector, measurement output vector, and input vector. $A_1(i,j)$, $A_2(i,j)$, $C(i,j)$, $D_k(i,j)$, $F_k(i,j)$, $(k=1,2,3)$ are time-varying matrices with appropriate dimensions. $d(i,j)\in R^s$, $v(i,j)\in R^v$ are the model disturbance and measurement disturbance, and they are zero mean independent white noises which satisfy:

$$\begin{aligned} &\left\langle \begin{bmatrix} d(i_1,j_1) \\ v(i_1,j_1) \end{bmatrix}, \begin{bmatrix} d(i_2,j_2) \\ v(i_2,j_2) \end{bmatrix} \right\rangle \\ &= \begin{bmatrix} R(i_1,j_1)\Delta_{(i_1,j_1),(i_2,j_2)} & 0 \\ 0 & Q(i_1,j_1)\Delta_{(i_1,j_1),(i_2,j_2)} \end{bmatrix} \end{aligned} \tag{2}$$

where $\langle m,n\rangle=\mathbb{E}(mn^T)$ and

$$\Delta_{(i_1,j_1),(i_2,j_2)}=\begin{cases} 1, for\, i_1=i_2\; and\; j_1=j_2. \\ 0, else. \end{cases} \tag{3}$$

As the fault description that given in 1-D systems [22] , the dynamic characteristics of $f(i+1,j+1)$ can be given as follows:

$$\begin{aligned} &f(i+1,j+1)=A_{f1}(i,j+1)f(i,j+1) \\ &+A_{f2}(i+1,j)f(i+1,j) \end{aligned} \tag{4}$$

where $A_{fk}(i,j)\,(k=1,2)$ are known matrices with appropriate dimensions.

Define $\bar{x}(i,j)=\begin{bmatrix} x(i,j)^T & f(i,j)^T \end{bmatrix}^T$ and combine (1) with (4), the augmented 2-D system can be obtained as follows:

$$\begin{cases} \bar{x}(i+1,j+1)=\bar{A}_1(i,j+1)\bar{x}(i,j+1) \\ +\bar{A}_2(i+1,j)\bar{x}(i+1,j)+\bar{D}_1(i,j+1)d(i,j+1) \\ +\bar{D}_2(i+1,j)d(i+1,j) \\ y(i,j)=\bar{C}(i,j)\bar{x}(i,j)+D_3(i,j)v(i,j) \end{cases} \tag{5}$$

where

$$\bar{A}_1(i,j+1)=\begin{bmatrix} A_1(i,j+1) & F_1(i,j+1) \\ \mathbf{0} & A_{f1}(i,j+1) \end{bmatrix},$$

$$\bar{A}_2(i+1,j)=\begin{bmatrix} A_2(i+1,j) & F_2(i+1,j) \\ \mathbf{0} & A_{f2}(i+1,j) \end{bmatrix},$$

$$\bar{D}_1(i,j+1)=\begin{bmatrix} D_1(i,j+1) \\ \mathbf{0} \end{bmatrix}, \bar{D}_2(i+1,j)=\begin{bmatrix} D_2(i+1,j) \\ \mathbf{0} \end{bmatrix},$$

$$\bar{C}(i,j)=\begin{bmatrix} C(i,j) & F_3(i,j) \end{bmatrix}$$

In this study, for the augmented 2-D system (5), the following estimator is considered:

$$\begin{cases} \hat{\bar{x}}(i+1,j+1)=z(i+1,j+1) \\ +J(i+1,j+1)y(i+1,j+1) \\ z(i+1,j+1)=G_1(i,j+1)z(i,j+1) \\ +G_2(i+1,j)z(i+1,j)+H_1(i,j+1)y(i,j+1) \\ +H_2(i+1,j)y(i+1,j) \end{cases} \tag{6}$$

where $\hat{\bar{x}}(i,j)\in R^{n+q}$ is the estimation of $\bar{x}(i,j)$, $z(i,j)\in R^l$ is the state of the estimation. Then, the fault and state estimation problem can be transformed to find estimator gain matrices $G_1(i,j+1)$, $G_2(i+1,j)$, $H_1(i,j+1)$, $H_2(i+1,j)$, and $J(i+1,j+1)$ to meet the definition of unbiased minimum-variance estimation.

The definition of unbiased minimum-variance estimation for the augmented 2-D system (5) can be presented as follows:

Definition 1. For the augmented 2-D system (5), $\hat{\bar{x}}(i,j)$ is the unbiased minimum-variance estimation of $\bar{x}(i,j)$, which satisfies:

$$\mathbb{E}\left\{\bar{x}(i,j)-\hat{\bar{x}}(i,j)\right\}=0 \tag{7}$$

$$\hat{\bar{x}}(i,j)=\arg\min\left\{trace\left\langle \bar{x}(i,j)-\hat{\bar{x}}(i,j),\bar{x}(i,j)-\hat{\bar{x}}(i,j)\right\rangle\right\} \tag{8}$$

(7) means the estimation is unbiased and (8) means the estimation has minimum variance.

III. Main Results

In this section, the unbiased minimum-variance estimator for augmented 2-D system will be obtained. In Theorem 1, the unbiased estimation of state and fault will be obtained. Based on the analysis of Theorem 1, minimum variance estimator will be further given in Theorem 2.

Lemma 1. Consider the augmented 2-D system (5) and assume that $\hat{\bar{x}}(i,j+1)$, $\hat{\bar{x}}(i+1,j)$ are unbiased estimation of $\bar{x}(i,j+1)$, $\bar{x}(i+1,j)$. If and only if

$$
\begin{aligned}
&S(i+1,j+1)\bar{A}_1(i,j+1)-H_1(i,j+1)\bar{C}(i,j+1)\\
&-G_1(i,j+1)S(i,j+1)=0\\
&S(i+1,j+1)\bar{A}_2(i+1,j)-H_2(i+1,j)\bar{C}(i+1,j)\\
&-G_2(i+1,j)S(i+1,j)=0
\end{aligned} \tag{9}
$$

holds, then, $\hat{\bar{x}}(i+1,j+1)$ is the unbiased estimation of $\bar{x}(i+1,j+1)$.

Proof. Define estimation error $e(i,j)=\bar{x}(i,j)-\hat{\bar{x}}(i,j)$, from (5) and (6), one can have

$$
\begin{aligned}
&\mathbb{E}\{e(i+1,j+1)\}=\mathbb{E}\{G_1(i,j+1)e(i,j+1)\\
&+G_2(i+1,j)e(i+1,j)+S(i+1,j+1)\bar{D}_1(i,j+1)d(i,j+1)\\
&+S(i+1,j+1)\bar{D}_2(i+1,j)d(i+1,j)\\
&-J(i+1,j+1)D_3(i+1,j+1)v(i+1,j+1)\\
&+\begin{bmatrix}S(i+1,j+1)\bar{A}_1(i,j+1)-H_1(i,j+1)\bar{C}(i,j+1)\\-G_1(i,j+1)S(i,j+1)\end{bmatrix}\bar{x}(i,j+1)\\
&+\begin{bmatrix}S(i+1,j+1)\bar{A}_2(i+1,j)-H_2(i+1,j)\bar{C}(i+1,j)\\-G_2(i+1,j)S(i+1,j)\end{bmatrix}\bar{x}(i+1,j)\\
&-H_1(i,j+1)D_3(i,j+1)v(i,j+1)\\
&+G_1(i,j+1)J(i,j+1)D_3(i,j+1)v(i,j+1)\\
&+G_2(i+1,j)J(i+1,j)D_3(i+1,j)v(i+1,j)\}
\end{aligned} \tag{10}
$$

where

$$
S(i,j)=\bar{I}-J(i,j)\bar{C}(i,j),\bar{I}=\begin{bmatrix}I&\mathbf{0}\\\mathbf{0}&I\end{bmatrix}
$$

Because $\hat{\bar{x}}(i,j+1)$, $\hat{\bar{x}}(i+1,j)$ are unbiased estimation of $\bar{x}(i,j+1)$, $\bar{x}(i+1,j)$, $\mathbb{E}\{e(i,j+1)\}=\mathbb{E}\{e(i+1,j)\}=0$, $d(i,j)$, $v(i,j)$ are white noise, $\mathbb{E}\{d(i,j)\}=\mathbb{E}\{v(i,j)\}=0$.

Then, if and only if (9) holds, one can get $\mathbb{E}\{e(i+1,j+1)\}=0$. It means that $\hat{\bar{x}}(i+1,j+1)$ is the unbiased estimation of $\bar{x}(i+1,j+1)$.

Define the matrices in (9) and estimator gain matrices as follows:

$$
\Xi(i+1,j+1)=\begin{bmatrix}-S(i,j+1)&0\\0&-S(i+1,j)\\-\bar{C}(i,j+1)&0\\0&-\bar{C}(i+1,j)\\-\bar{C}(i+1,j+1)\bar{A}_1(i,j+1)&-\bar{C}(i+1,j+1)\bar{A}_2(i+1,j)\end{bmatrix}
$$

$$
\Phi(i+1,j+1)=\begin{bmatrix}-\bar{A}_1(i,j+1)&-\bar{A}_2(i+1,j)\end{bmatrix}
$$

$$
\begin{aligned}
\mathrm{X}(i+1,j+1)=[&G_1(i,j+1)\quad G_2(i+1,j)\\
&H_1(i,j+1)\quad H_2(i+1,j)\quad J(i+1,j+1)]
\end{aligned}
$$

According to (9), the following matrix equation can be obtained:

$$
\mathrm{X}(i+1,j+1)\Xi(i+1,j+1)=\Phi(i+1,j+1) \tag{11}
$$

It can be rewritten as:

$$
\Xi(i+1,j+1)^T\mathrm{X}(i+1,j+1)^T=\Phi(i+1,j+1)^T
$$

To solve this matrix equation, If and only if:

$$
rank\left(\Xi(i+1,j+1)^T\right)=rank\begin{bmatrix}\Xi(i+1,j+1)\\\Phi(i+1,j+1)\end{bmatrix}^T \tag{12}
$$

holds, the matrix equation (11) has solutions.

Next, we need to analysis the rank of $\Xi(i+1,j+1)^T$ and $\begin{bmatrix}\Xi(i+1,j+1)^T & \Phi(i+1,j+1)^T\end{bmatrix}$. (12) equals:

$$
rank\left(\Xi(i+1,j+1)\right)=rank\begin{bmatrix}\Xi(i+1,j+1)\\\Phi(i+1,j+1)\end{bmatrix} \tag{13}
$$

According to some simple matrix transformation, one can get:

$$
\begin{aligned}
&rank\left(\Xi(i+1,j+1)\right)\\
&=rank\begin{bmatrix}-\bar{I}&0\\0&-\bar{I}\\-\bar{C}(i,j+1)&0\\0&-\bar{C}(i+1,j)\\0&0\end{bmatrix}=rank\begin{bmatrix}-\bar{I}&0\\0&-\bar{I}\end{bmatrix}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&rank\begin{bmatrix}\Xi(i+1,j+1)\\\Phi(i+1,j+1)\end{bmatrix}\\
&=rank\begin{bmatrix}-\bar{I}&0\\0&-\bar{I}\\-\bar{C}(i,j+1)&0\\0&-\bar{C}(i+1,j)\\0&0\\-\bar{A}_1(i,j+1)&-\bar{A}_2(i+1,j)\end{bmatrix}=rank\begin{bmatrix}-\bar{I}&0\\0&-\bar{I}\end{bmatrix}
\end{aligned}
$$

Therefore, (12) holds, for augmented 2-D system (5), by solving matrix (11), one can get the unbiased estimator gain matrices.

Theorem 1: If and only if (9) holds, estimator (6) is unbiased estimator for system (5), and the unbiased estimator gain matrices can be obtained as follows:

$$
\mathrm{X}(i+1,j+1)=a+\Pi b \tag{14}
$$

where

$$
\begin{aligned}
&a=\Phi(i+1,j+1)\Xi^+(i+1,j+1),\\
&b=\left(I-\Xi(i+1,j+1)\Xi^+(i+1,j+1)\right),
\end{aligned}
$$

Π is an arbitrary matrix with appropriate dimension.

Proof: Referring to the previous analysis of solvability of matrix equation (11), the general solution of (11) can be obtained in (14) . It completes the proof.

Now, the solution of unbiased estimation has been obtained. To get a further result of the minimum-variance state and fault estimation, one needs to choose an appropriate Π to minimize the variance of the estimation error under the solution of (14).

Theorem 2. Suppose that (9) and (14) hold, the estimator has unbiased minimum-variance solution, if and only if:

$$\begin{aligned}\Pi &= \Phi_\Pi(i+1,j+1)\Xi_\Pi^{+}(i+1,j+1)\\ &+\Omega\left(I-\Xi_\Pi(i+1,j+1)\Xi_\Pi^{+}(i+1,j+1)\right)\end{aligned} \tag{15}$$

holds, Ω is an arbitrary matrix with appropriate dimension. Φ_Π and Ξ_Π are defined in (21) and (22), respectively.

The desired estimator gain matrices can be obtained as follows:

$$\begin{aligned}\mathrm{X}(i+1,j+1) &= a+\Pi b\\ &= a+\begin{bmatrix}\Phi_\Pi(i+1,j+1)\Xi_\Pi^{+}(i+1,j+1)\\ +\Omega\left(I-\Xi_\Pi(i+1,j+1)\Xi_\Pi^{+}(i+1,j+1)\right)\end{bmatrix}b\end{aligned} \tag{16}$$

Proof. According to (10), we rewrite the estimation error as:

$$\begin{aligned}&\mathbb{E}\left\{e(i+1,j+1)\right\}=\\ &\mathbb{E}\left\{\mathrm{X}(i+1,j+1)\vartheta(i+1,j+1)+\phi(i+1,j+1)\right\}\end{aligned} \tag{17}$$

where

$$\vartheta(i+1,j+1)=\begin{bmatrix}e(i,j+1)-J(i,j+1)D_3(i,j+1)v(i,j+1)\\ e(i+1,j)-J(i+1,j)D_3(i+1,j)v(i+1,j)\\ -D_3(i,j+1)v(i,j+1)\\ -D_3(i+1,j)v(i+1,j)\\ -D_3(i+1,j+1)v(i+1,j+1)-\bar{C}(i+1,j+1)\phi(i+1,j+1)\end{bmatrix}$$

$$\phi(i+1,j+1)=\bar{D}_1(i,j+1)d(i,j+1)+\bar{D}_2(i+1,j)d(i+1,j)$$

Define $P(i+1,j+1)$ as the covariance matrix of the estimation error, it can be expressed as:

$$\begin{aligned}P(i+1,j+1) &= \mathbb{E}\left[e(i+1,j+1)e^T(i+1,j+1)\right]\\ &= \mathbb{E}\begin{bmatrix}\left(\mathrm{X}(i+1,j+1)\vartheta(i+1,j+1)+\phi(i+1,j+1)\right)*\\ \left(\mathrm{X}(i+1,j+1)\vartheta(i+1,j+1)+\phi(i+1,j+1)\right)^T\end{bmatrix}\\ &= \mathbb{E}\begin{bmatrix}\mathrm{X}(i+1,j+1)\left\langle\vartheta(i+1,j+1),\vartheta(i+1,j+1)\right\rangle *\\ \mathrm{X}(i+1,j+1)^T+\left\langle\phi(i+1,j+1),\phi(i+1,j+1)\right\rangle\\ +\mathrm{X}(i+1,j+1)\left\langle\vartheta(i+1,j+1),\phi(i+1,j+1)\right\rangle\\ +\left\langle\phi(i+1,j+1),\vartheta(i+1,j+1)\right\rangle\mathrm{X}(i+1,j+1)^T\end{bmatrix}\end{aligned} \tag{18}$$

To obtain the minimum-variance estimator, we need to derivate the functions of $P(i+1,j+1)$ with respect to $\mathrm{X}(i+1,j+1)$. Based on the theory of matrix derivation, the derivation of *trace* $(P(i+1,j+1))$ with respect to Π can be obtained:

$$\begin{Bmatrix}2X(i+1,j+1)*\left\langle\vartheta(i+1,j+1),\vartheta(i+1,j+1)\right\rangle\\ +2\left\langle\phi(i+1,j+1),\vartheta(i+1,j+1)\right\rangle\end{Bmatrix}b^T \tag{19}$$

Let (19) equal 0, we have the following matrix equation:

$$\Pi\Xi_\Pi(i+1,j+1)=\Phi_\Pi(i+1,j+1) \tag{20}$$

where

$$\begin{aligned}&\Xi_\Pi(i+1,j+1)=b\left\langle\vartheta(i+1,j+1),\vartheta(i+1,j+1)\right\rangle b^T\\ &=b\begin{bmatrix}\Xi_{\Pi(11)} & \Xi_{\Pi(12)} & \Xi_{\Pi(13)} & 0 & 0\\ * & \Xi_{\Pi(22)} & 0 & \Xi_{\Pi(24)} & 0\\ * & * & \Xi_{\Pi(33)} & 0 & 0\\ * & * & * & \Xi_{\Pi(44)} & 0\\ * & * & * & * & \Xi_{\Pi(55)}\end{bmatrix}b^T\\ &\Xi_{\Pi(11)}=P(i,j+1)+J(i,j+1)\\ &*D_3(i,j+1)Q(i,j+1)D_3^T(i,j+1)J^T(i,j+1)\\ &\Xi_{\Pi(12)}=\left\langle e(i,j+1),e(i+1,j)\right\rangle\\ &\Xi_{\Pi(13)}=-J(i,j+1)D_3(i,j+1)Q(i,j+1)D_3^T(i,j+1)\\ &\Xi_{\Pi(22)}=P(i+1,j)+J(i+1,j)\\ &*D_3(i+1,j)Q(i+1,j)D_3^T(i+1,j)J^T(i+1,j)\\ &\Xi_{\Pi(24)}=-J(i+1,j)D_3(i+1,j)Q(i+1,j)D_3^T(i+1,j)\\ &\Xi_{\Pi(33)}=D_3(i,j+1)Q(i,j+1)D_3^T(i,j+1)\\ &\Xi_{\Pi(44)}=D_3(i+1,j)Q(i+1,j)D_3^T(i+1,j)\\ &\Xi_{\Pi(55)}=D_3(i+1,j+1)Q(i+1,j+1)D_3^T(i+1,j+1)\\ &+\bar{C}(i+1,j+1)\left[\bar{D}_1(i,j+1)R(i,j+1)\bar{D}_1^T(i,j+1)+\right.\\ &\left.\bar{D}_2(i+1,j)R(i+1,j)\bar{D}_2^T(i+1,j)\right]\bar{C}^T(i+1,j+1)\end{aligned} \tag{21}$$

and

$$
\Phi_{\Pi}\left(i+1,j+1\right)=-\begin{Bmatrix} a\left\langle \vartheta(i+1,j+1),\vartheta(i+1,j+1)\right\rangle \\ +\left\langle \phi(i+1,j+1),\vartheta(i+1,j+1)\right\rangle \end{Bmatrix} b^{T}
$$

$$
=-a\begin{bmatrix} \Xi_{\Pi(11)} & \Xi_{\Pi(12)} & \Xi_{\Pi(13)} & 0 & 0 \\ * & \Xi_{\Pi(22)} & 0 & \Xi_{\Pi(24)} & 0 \\ * & * & \Xi_{\Pi(33)} & 0 & 0 \\ * & * & * & \Xi_{\Pi(44)} & 0 \\ * & * & * & * & \Xi_{\Pi(55)} \end{bmatrix} b^{T}
$$

$$
+\left[0 \quad 0 \quad 0 \quad 0 \quad \Xi_{\Pi 5}\right] b^{T}
$$

$$
\Xi_{\Pi 5}=\begin{bmatrix} \bar{D}_{1}\left(i,j+1\right)R\left(i,j+1\right)\bar{D}_{1}^{T}\left(i,j+1\right) \\ +\bar{D}_{2}\left(i+1,j\right)R\left(i+1,j\right)\bar{D}_{2}^{T}\left(i+1,j\right) \end{bmatrix}
$$

$$
*\bar{C}^{T}\left(i+1,j+1\right) \tag{22}
$$

By solving matrix (20), we can get the appropriate Π as:

$$
\begin{aligned} \Pi &= \Phi_{\Pi}\left(i+1,j+1\right)\Xi_{\Pi}^{+}(i+1,j+1) \\ &+\Omega\left(I-\Xi_{\Pi}(i+1,j+1)\Xi_{\Pi}^{+}(i+1,j+1)\right) \end{aligned}
$$

Substituting the solution of Π into (14), the unbiased minimum-variance estimator gain matrices can be deduced in (16). It completes the proof.

IV. Illustrative Example

In this section, some examples are provided to demonstrate the feasibility and effectiveness of the proposed observer design method in the previous theorems.

Consider the 2-D FM-II system, where the system coefficient matrices and fault vectors are given as follows:

$$
A_{1}(i,j)=\begin{bmatrix} -0.1 & 1 \\ 0 & 0.1+0.1\sin(i+j) \end{bmatrix}
$$

$$
A_{2}(i,j)=\begin{bmatrix} 0.1+0.1\sin(i+j) & 0 \\ 0 & 0.1 \end{bmatrix},\quad C(i,j)=\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
D_{1}(i,j)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, D_{2}(i,j)=\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, D_{3}(i,j)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
F_{1}(i,j)=\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad F_{2}(i,j)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, F_{3}(i,j)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

$$
f(i,j)=0.5f(i-1,j)+0.5f\left(i,j-1\right)
$$

The model noise and measurement noise can be described as:

$$
w(i,j)\sim N(0,0.1), v(i,j)\sim N(0,0.03).
$$

Consider a time window $(0,0)\le(i,j)\le(50,50)$ and suppose the initial value of fault as:

$$
f(i,0)=\begin{cases} 1, for\, i>25 \\ 0, else \end{cases},\quad f\left(0,j\right)=\begin{cases} 1, for\ j>1 \\ 0, else \end{cases}
$$

According to the designed estimator in (16), the state and fault estimation results are given in Figs. 1, 2, and 3. Figs. 1 and 2 show system state estimation, and Fig. 3 presents the fault estimation. In these figures, the left parts are the true values and the right parts are the corresponding estimations.

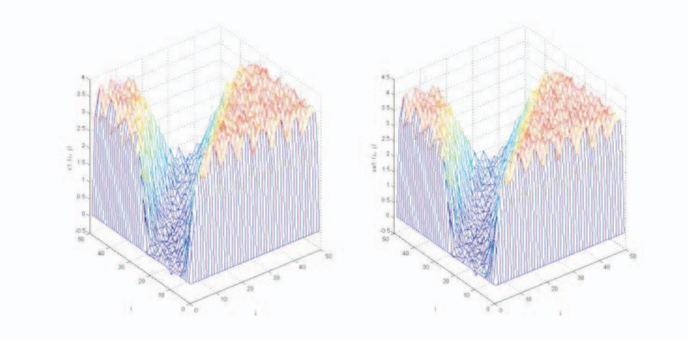

Fig. 1. System state $x_1(i,j)$ and its estimation. Left: true value. Right: estimation.

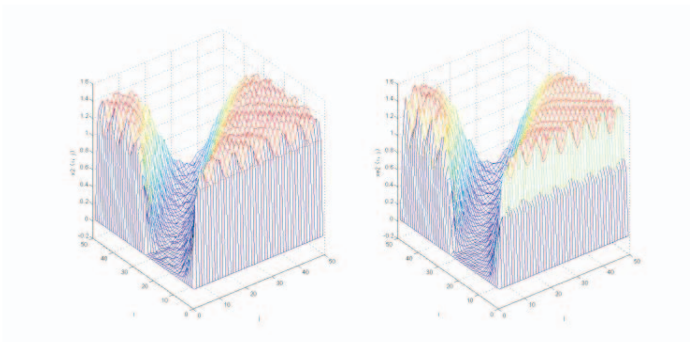

Fig. 2. System state $x_2(i,j)$ and its estimation. Left: true value. Right: estimation.

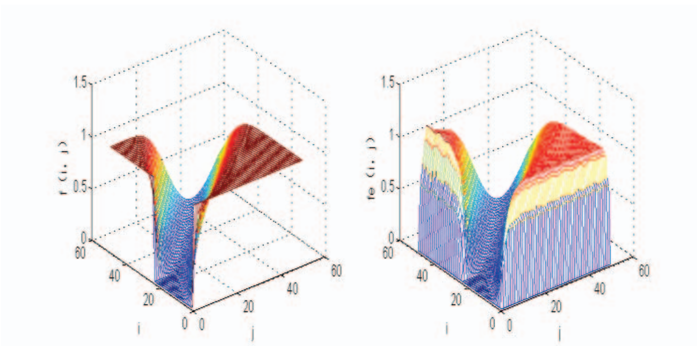

Fig. 3. System fault $f(i,j)$ and its estimation. Left: true value. Right: estimation.

It can be found that, for the 2-D time-varying systems subject to white noise and fault, the proposed estimator simultaneously products accurate estimation for system state and fault in the sense of unbiased minimum-variance.

V. CONCLUSIONS

The fault and state estimation problem of 2-D time-varying systems are investigated in this study. Under the effects of fault and disturbance, unbiased minimum variance estimator has been obtained. The estimation results can be used for fault reconstruction and active fault tolerant control. In the future, 2-D nonlinear time-varying systems will be discussed.

REFERENCES

[1] E. Fornasini and G. Marchesini, "State-space realization theory of two-dimensional filters," *IEEE Transactions on Automatic Control,* vol. 21, pp. 484-492, 1976.

[2] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Transactions on Automatic Control,* vol. 20, pp. 1-10, 1975.

[3] D. Zhao, Z. Lin, and Y. Wang, "Integrated state/disturbance observers for two-dimensional linear systems," *Iet Control Theory & Applications,* vol. 9, pp. 1373-1383, 2015.

[4] M. Bisiacco and M. E. Valcher, "Observer-based fault detection and isolation for 2D state-space models," *Multidimensional Systems and Signal Processing,* vol. 17, pp. 219-242, 2006.

[5] C. E. De Souza and J. Osowsky, "Gain-scheduled control of two-dimensional discrete-time linear parameter-varying systems in the Roesser model," *Automatica,* vol. 49, pp. 101-110, 2013.

[6] C. Du, L. Xie, and C. Zhang, "H∞ control and robust stabilization of two-dimensional systems in Roesser models," *Automatica,* vol. 37, pp. 205-211, 2001.

[7] J. Liang, Z. Wang, Y. Liu, and X. Liu, "State estimation for two-dimensional complex networks with randomly occurring nonlinearities and randomly varying sensor delays," *International Journal of Robust and Nonlinear Control,* vol. 24, pp. 18-38, 2014.

[8] M. B. Sumanasena and P. H. Bauer, "Realization using the Fornasini-Marchesini model for implementations in distributed grid sensor networks," *IEEE Transactions on Circuits and Systems I: Regular Papers,* vol. 58, pp. 2708-2717, 2011.

[9] T. Hinamoto, "2-D Lyapunov equation and filter design based on the Fornasini-Marchesini second model," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,* vol. 40, pp. 102-110, 1993.

[10] Y. Wang, D. Zhou, and F. Gao, "Iterative learning reliable control of batch processes with sensor faults," *Chemical Engineering Science,* vol. 63, pp. 1039-1051, 2008.

[11] S. X. Ding, "Integrated design of feedback controllers and fault detectors," *Annual Reviews in Control,* vol. 33, pp. 124-135, 2009.

[12] Z. Gao, "Fault estimation and fault tolerant control for discrete-time dynamic systems," *IEEE Transactions on Industrial Electronics,* vol. 62, pp. 3874-3884, 2015.

[13] Z. Gao, X. Liu, and M. Chen, "Unknown input observer based robust fault estimation for systems corrupted by partially decoupled disturbances," *IEEE Transactions on Industrial Electronics,* vol. 63, pp. 2537-2547, 2016.

[14] D. Zhao, D. Shen, and Y. Wang, "Fault diagnosis and compensation for two-dimensional discrete time systems with sensor faults and time-varying delays," *International Journal of Robust and Nonlinear Control,* 2017, unpublished.

[15] L. Cao, Y. Tao, Y. Wang, J. Li, and B. Huang, "Reliable H∞ control for nonlinear discrete-time systems with multiple intermittent faults in sensors or actuators," *International Journal of Systems Science,* pp. 1-14, 2016.

[16] M. Liu, P. Shi, L. Zhang, and X. Zhao, "Fault-tolerant control for nonlinear Markovian jump systems via proportional and derivative sliding mode observer technique," *IEEE Transactions on Circuits and Systems I: Regular Papers,* vol. 58, pp. 2755-2764, 2011.

[17] Z. Wang and H. Shang, "Kalman filter based fault detection for two-dimensional systems," *Journal of Process Control,* vol. 28, pp. 83-94, 2015.

[18] D. Ding, H. Wang, and X. Li, "H−/H∞ fault detection observer design for two-dimensional Roesser systems," *Systems & Control Letters,* vol. 82, pp. 115-120, 2015.

[19] D. Zhao and Y. Wang, "Integrated Fault Detection for Two-Dimensional Discrete Systems with Time-Varying Delays," *IFAC International Symposium on Fault Detection,Supervision and Safety for Technical Process,* vol. 48, pp. 1262-1267, 2015.

[20] X. Li and K. Zhou, "A time domain approach to robust fault detection of linear time-varying systems," *Automatica,* vol. 45, pp. 94-102, 2009.

[21] M. Zhong, S. X. Ding, J. Lam, and H. Wang, "An LMI approach to design robust fault detection filter for uncertain LTI systems," *Automatica,* vol. 39, pp. 543-550, 2003.

[22] H. Dong, Z. Wang, S. X. Ding, and H. Gao, "Finite-horizon estimation of randomly occurring faults for a class of nonlinear time-varying systems," *Automatica,* vol. 50, pp. 3182-3189, 2014.

[23] Y. Zou, M. Sheng, N. Zhong, and S. Xu, "A generalized Kalman filter for 2D discrete systems," *Circuits, Systems and Signal Processing,* vol. 23, pp. 351-364, 2004.

[24] R. Yang, L. Ntogramatzidis, and M. Cantoni, "On Kalman filtering for 2-D Fornasini-Marchesini models," in *IEEE Multidimensional (nD) Systems International Workshop, Thessaloniki*, 2009