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Existence and design of observers for two-dimensional linear systems with multiple channel faults

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Abstract Integrated states/faults observers for two-dimensional (2-D) linear systems, which can simultaneously estimate system states and faults, are studied in this study. Multiple channel faults, occurring in both measurement equation and state equation, are considered. On the basis of the singular system observer method and stability theory, asymptotically stable observers and uniformly ultimately bounded observers are proposed for the 2-D systems. For these two cases, constructive methods for observer design and parameter tuning are further provided. Under different system conditions, the necessary and sufficient conditions for the existence of integrated observers are derived and proved through matrix rank analysis. Finally, two examples are given to demonstrate the performance of the proposed methods.

Keywords States/faults estimation · Singular system observer · Two-dimensional systems · Asymptotically stable observer · Uniformly ultimately bounded observer

1 Introduction

The exploration of two-dimensional (2-D) systems dates back to the 1970s (Roesser 1975) and such systems have widespread applications in X-ray image enhancement (Kaczorek 1985), digital picture processing (Xie et al. 2002; Lin and Bruton 1989), thermal processes (Marszalek 1984), and batch process analysis (Wang et al. 2018). Among these research

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areas, existence and design of observers in the context of 2-D systems have received extensive attention in the past several decades.

Various 2-D observers and design methods have been proposed and analyzed, including dead-beat observer (Bisiacco and Valcher 2008; Wang and Shang 2015), Luenberger observer (Bisiacco and Valcher 2007), unknown input observer (Zhao et al. 2015; Bhattacharyya 1978; Ntogramatzidis and Cantoni 2012; Ntogramatzidis 2007), functional observer (Xu et al. 2012) nonlinear observer (Zhao et al. 2017; Ghous and Xiang 2015; Liang et al. 2014a, b), and singular system observer (Zou et al. 2007; Wang and Zou 2002). Moreover, among the existed results on 2-D observer, some could be further detailed, like the fundamental studies on 2-D observers with necessary and sufficient conditions for state reconstruction and the structure of observers for 2-D systems based on polynomial matrix theory (Bisiacco 1986), and the 2-D nonlinear functional observer with existence condition in terms of matrix rank (Wang et al. 2013).

To explore deeper, for multidimensional systems, a geometric fault detection and isolation (FDI) approach for discrete-time multidimensional systems is proposed in Baniamerian et al. (2016). Using basic invariant subspaces including unobservable, conditioned invariant and unobservability subspaces for multidimensional models, necessary and sufficient conditions for detectability and isolability of faults are provided. In Maleki et al. (2015), the basic results are introduced in Theorems 2 and 3, which gives necessary conditions for fault detection and sufficient conditions for the existence of an observer. In Ntogramatzidis and Cantoni (2012), it leads to an LMI-based procedure for the synthesis of observers that asymptotically estimate the local state of a standard Fornasini–Marchesini model.

Although creative studies have been conducted on the existence and design of observers for multidimensional systems (including 2-D system as a special case), their results mostly present sufficient or necessary conditions for the existence of observers (Baniamerian et al. 2016; Ntogramatzidis and Cantoni 2012; Wang et al. 2018) and rarely consider the estimation of multiple channel faults (Wang et al. 2017). These results also did not consider the case that there may be no asymptotically stable observers for multidimensional systems. For 2-D systems with multiple faults, no reported methods currently can estimate states and multiple faults simultaneously. However, simultaneous estimation of system states and faults are significant for fault diagnosis and fault compensation control. For example, if the observers can detect, locate, and estimate the occurring multiple faults at the same time, then the fault identification and signal compensation can be further performed rather than merely detected and isolated faults. For one-dimensional (1-D) systems, the problem about fault observers has attracted extensive attention and many results have been published (Dong et al. 2008; Gao and Ding 2007; Wang et al. 2009; Lin and Liu 2006; Koenig 2005; Massoumnia 1986). However, both problem description and system properties for the 2-D case are markedly different from that for the 1-D case. In this case, we cannot just follow the same technique line for 1-D problem to obtain the expected solution for a 2-D case.

Inspired by the singular system method for 1-D case (Gao and Wang 2006), we propose the corresponding 2-D fault observers. In this study, the 2-D system described by the Fornasini and Marchesini second (FM-II) model (Fornasini and Marchesini 1976, 1978) is considered. Under different system conditions, we first provide the necessary and sufficient conditions on whether or not the states and faults can be estimated.

The rest of the paper is organized as follows. In Sect. 2, some definitions and lemmas are presented for the considered systems. Section 3 provides both existence conditions and solution for the asymptotically stable observers and uniformly ultimately bounded observers. In Sect. 4, the effectiveness of the proposed method is validated by simulation examples. Section 5 summarizes the results of this study.

2 Problem formulation and preliminary knowledge

The following FM-II model is used to describe a 2-D system with faults in both the measurement equation and the state equation:

$$\begin{cases} x(i+1, j+1) = A_1 x(i, j+1) + A_2 x(i+1, j) + B_1 u(i, j+1) + B_2 u(i+1, j) \\ \qquad\qquad + M_1 f(i, j+1) + M_2 f(i+1, j) \\ y(i, j) = Cx(i, j) + f_s(i, j) \end{cases} \tag{1}$$

and unknown boundary conditions satisfy:

$$\sup \|x(i, 0)\| < \infty, \quad i = 0, 1, \ldots; \quad \sup \|x(0, j)\| < \infty, \quad j = 1, 2, \ldots \tag{2}$$

where $x(i, j) \in R^n$, $y(i, j) \in R^p$, and $u(i, j) \in R^m$ are the system state vector, the output measurement vector, and the input vector, respectively. $f(i, j) \in R^q$ is the fault to represent actuator fault or process fault. $f_s(i, j) \in R^p$ is the sensor fault. $C, A_k, B_k, M_k (k = 1, 2)$ are known system matrices with appropriate dimensions.

Referring to the fault model proposed for 1-D systems (Dong et al. 2014), the dynamic characteristics of $f(i, j) \in R^q$ in 2-D systems can be modelled as follows:

$$f(i+1, j+1) = A_{d1} f(i, j+1) + A_{d2} f(i+1, j) \tag{3}$$

where A_{d1}, A_{d2} are known matrices with appropriate dimensions.

Remark 1 In this study, although the dynamic characteristics of $f(i, j)$ are described as (3), the fault trajectory is unknown because of the undetermined initial value. In practice, by analyzing historical faults data with identification, or regression methods, or using established fault dynamic characteristics of some common faults like motor fault, prior knowledge of fault can be obtained and it will simplify our analysis. Please see Dong et al. (2014) and references therein. For example, if we know the 2-D system fault is constant with unknown initial value, the fault can be described as follows:

$$f(i+1, j+1) = 0.5 f(i, j+1) + 0.5 f(i+1, j)$$

The objective of this study is to design observers to estimate system states and faults in both the measurement equation and the state equation simultaneously. Define $\bar{x}(i+1, j+1) = \left[x^T(i+1, j+1) \ f_s^T(i+1, j+1) \right]^T$ and consider the following observer:

$$\begin{cases} z(i+1, j+1) = F_1 z(i, j+1) + F_2 z(i+1, j) + H_1 u(i, j+1) + H_2 u(i+1, j) \\ \qquad\qquad + G_1 y(i, j+1) + G_2 y(i+1, j) + R_1 \hat{f}(i, j+1) + R_2 \hat{f}(i+1, j) \\ \hat{f}(i+1, j+1) = A_{d1} \hat{f}(i, j+1) + A_{d2} \hat{f}(i+1, j) \\ \qquad\qquad - K_{d1} \bar{C} z(i, j+1) - K_{d2} \bar{C} z(i+1, j) \\ \hat{\bar{x}}(i, j) = z(i, j) + Ty(i, j) \end{cases} \tag{4}$$

and unknown boundary conditions satisfy:

$$\sup \|z(i, 0)\| < \infty, \quad i = 0, 1, \ldots; \quad \sup \|z(0, j)\| < \infty, \quad j = 1, 2, \ldots \tag{5}$$

where, $z(i, j) \in R^{n+p}$ and $\hat{\bar{x}}(i, j) \in R^{n+p}$ are the state and output of the observer, respectively; $\hat{\bar{x}}(i, j)$ and $\hat{f}(i, j)$ are the estimation of $\bar{x}(i, j)$ and $f(i, j)$, respectively. F_1, F_2, H_1, H_2, G_1, G_2, R_1, R_2, K_{d1}, K_{d2} and T are the matrices to be determined.

Define $e(i, j) = \bar{x}(i, j) - \hat{\bar{x}}(i, j)$, $e_d(i, j) = f(i, j) - \hat{f}(i, j)$, and the following set: $\bar{U}^2 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \leq 1, |z_2| \leq 1\}$. Some essential definitions and lemmas for the 2-D linear system can be applied:

Definition 1 (Bose 1982) Assume that $F(z_1, z_2)$ is the Z transformation of $f(i, j)$, then the 2-D Z transformation formula is:

$$F(z_1, z_2) = \mathbf{Z}[f(i, j)] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i, j) z_1^i z_2^j \tag{6}$$

Definition 2 Observer (4) is an asymptotic observer for system (1), if $\lim_{i,j\to\infty} e(i, j) = \mathbf{0}$ and $\lim_{i,j\to\infty} e_d(i, j) = \mathbf{0}$ hold for any system boundary conditions satisfying (2), any observer boundary conditions satisfying (5), and any input sequence $u(i, j)$.

Definition 3 Observer (4) is a uniformly ultimately bounded observer for system (1), if there exist $\zeta > 0$, $\zeta_d > 0$, $I > 0$, $J > 0$ such that $\|e(i, j)\| \leq \zeta$ and $\|e_d(i, j)\| \leq \zeta_d$ $(\forall i \geq I, \forall j \geq J)$ holds for any system boundary conditions satisfying (2), any observer boundary conditions satisfying (5), any input sequence $u(i, j)$.

Lemma 1 (Fornasini and Marchesini 1980) *The 2-D system* (1) *is open-loop asymptotically stable or* (A_1, A_2) *is an asymptotically stable pair, if and only if the following rank condition holds*.

$$rank(I_n - z_1 A_1 - z_2 A_2) = n, \quad \forall (z_1, z_2) \in \bar{U}^2$$

Lemma 2 (Bisiacco 1986) *The 2-D system* (1) *is detectable or* (A_1, A_2, C) *is a detectable triplet, if and only if the following rank condition holds*.

$$rank \begin{bmatrix} C \\ I_n - z_1 A_1 - z_2 A_2 \end{bmatrix} = n, \quad \forall (z_1, z_2) \in \bar{U}^2$$

Lemma 3 (Bose 1982) *Assuming* $F(z_1, z_2)$ *is the Z transformation of* $f(i, j)$, *then the 2-D final-value theorem can be given as follows*:

$$\lim_{z_1\to 1, z_2\to 1} \left\{ \frac{(1 - z_1)(1 - z_2)}{z_1 z_2} F(z_1, z_2) \right\} = \lim_{i\to\infty, j\to\infty} \{f(i, j)\} \tag{7}$$

3 Observer designs

3.1 Asymptotically stable observer

Define the following matrices:

$$E = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} A_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix},$$
$$\bar{B}_2 = \begin{bmatrix} B_2 \\ \mathbf{0} \end{bmatrix}, \bar{C} = \begin{bmatrix} C & I_p \end{bmatrix}, \bar{M}_1 = \begin{bmatrix} M_1 \\ \mathbf{0} \end{bmatrix}, \bar{M}_2 = \begin{bmatrix} M_2 \\ \mathbf{0} \end{bmatrix} \tag{8}$$

One can obtain the following augmented 2-D singular system from system (1):

$$\begin{cases} E\bar{x}(i+1, j+1) = \bar{A}_1\bar{x}(i, j+1) + \bar{A}_2\bar{x}(i+1, j) + \bar{B}_1 u(i, j+1) + \bar{B}_2 u(i+1, j) \\ \qquad\qquad + \bar{M}_1 f(i, j+1) + \bar{M}_2 f(i+1, j) \\ y(i, j) = \bar{C}\bar{x}(i, j) \\ f(i+1, j+1) = A_{d1} f(i, j+1) + A_{d2} f(i+1, j) \end{cases} \tag{9}$$

and unknown boundary conditions satisfy:

$$\begin{aligned} &\sup \|\bar{x}(i,0)\| < \infty, \quad i = 0, 1, \ldots; \quad \sup \|\bar{x}(0,j)\| < \infty, \quad j = 1, 2, \ldots \\ &\sup \|f(i,0)\| < \infty, \quad i = 0, 1, \ldots; \quad \sup \|f(0,j)\| < \infty, \quad j = 1, 2, \ldots \end{aligned} \tag{10}$$

Theorem 1 *For the 2-D singular system* (9), *with boundary conditions* (10), *if and only if*:

$$\begin{cases} rank(I_n - z_1 A_1 - z_2 A_2) = n \\ rank(I_q - z_1 A_{d1} - z_2 A_{d2}) = q \end{cases} \quad \forall (z_1, z_2) \in \bar{U}^2 \tag{11}$$

holds, then there exist the following observer matrices:

$$\begin{aligned} &L, K_1, K_2 \in R^{(n+p)\times p}, K_{d1}, K_{d2} \in R^{q\times p}, \\ &F_1 = (E + L\bar{C})^{-1}(\bar{A}_1 - K_1\bar{C}), F_2 = (E + L\bar{C})^{-1}(\bar{A}_2 - K_2\bar{C}), \\ &H_1 = (E + L\bar{C})^{-1}\bar{B}_1, H_2 = (E + L\bar{C})^{-1}\bar{B}_2, T = (E + L\bar{C})^{-1}L, \\ &R_1 = (E + L\bar{C})^{-1}\bar{M}_1, R_2 = (E + L\bar{C})^{-1}\bar{M}_2, \\ &G_1 = (E + L\bar{C})^{-1}\bar{A}_1(E + L\bar{C})^{-1}L, G_2 = (E + L\bar{C})^{-1}\bar{A}_2(E + L\bar{C})^{-1}L \end{aligned} \tag{12}$$

and boundary conditions

$$\begin{aligned} &\sup \left\|\hat{\bar{x}}(i,0)\right\| < \infty, \quad i = 0, 1, \ldots; \quad \sup \left\|\hat{\bar{x}}(0,j)\right\| < \infty, \quad j = 1, 2, \ldots \\ &\sup \left\|\hat{f}(i,0)\right\| < \infty, \quad i = 0, 1, \ldots; \quad \sup \left\|\hat{f}(0,j)\right\| < \infty, \quad j = 1, 2, \ldots \end{aligned} \tag{13}$$

which can lead to $\lim\limits_{i,j\to\infty} (\bar{x}(i,j) - \hat{\bar{x}}(i,j)) = \mathbf{0}$ *and* $\lim\limits_{i,j\to\infty} (f(i,j) - \hat{f}(i,j)) = \mathbf{0}$.

Proof Partitioning $L = \begin{bmatrix} L_1^T & L_2^T \end{bmatrix}^T$, where $L_1 \in R^{n\times p}$,$L_2 \in R^{p\times p}$, note that

$$\begin{aligned} rank(E + L\bar{C}) &= rank\left(\begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} C & I_p \end{bmatrix}\right) \\ &= rank\begin{bmatrix} I_n + L_1 C & L_1 \\ L_2 C & L_2 \end{bmatrix} = rank\begin{bmatrix} I_n & L_1 \\ \mathbf{0} & L_2 \end{bmatrix} = rank(L_2) + n \end{aligned} \tag{14}$$

Hence, if and only if the square matrix L_2 is of full rank, $(E + L\bar{C})$ is nonsingular. Letting L_2 be nonsingular, the following relationship can be obtained:

$$\begin{aligned} \bar{C}(E + L\bar{C})^{-1}L &= \begin{bmatrix} C & I_p \end{bmatrix}\begin{bmatrix} I_n & -L_1(L_2)^{-1} \\ -C & (I_p + CL_1)(L_2)^{-1} \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \\ &= \begin{bmatrix} C & I_p \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ I_p \end{bmatrix} = I_p \end{aligned} \tag{15}$$

Adding $Ly(i+1, j+1)$ to both sides of the first equation in (9), it yields

$$\begin{aligned} (E + L\bar{C})\bar{x}(i+1, j+1) &= \bar{A}_1\bar{x}(i, j+1) + \bar{A}_2\bar{x}(i+1, j) + \bar{B}_1 u(i, j+1) + \bar{B}_2 u(i+1, j) \\ &\quad + \bar{M}_1 f(i, j+1) + \bar{M}_2 f(i+1, j) + Ly(i+1, j+1) \\ &= (\bar{A}_1 - K_1\bar{C})\bar{x}(i, j+1) + (\bar{A}_2 - K_2\bar{C})\bar{x}(i+1, j) \\ &\quad + K_1 y(i, j+1) + K_2 y(i+1, j) + \bar{B}_1 u(i, j+1) + \bar{B}_2 u(i+1, j) \\ &\quad + Ly(i+1, j+1) + \bar{M}_1 f(i, j+1) + \bar{M}_2 f(i+1, j) \end{aligned} \tag{16}$$

Substituting $z(i, j) = \widehat{\hat{x}}(i, j) - (E + L\bar{C})^{-1}Ly(i, j)$ into the observer shown in (4) and using Eq. (12), the observer in (4) becomes:

$$\begin{cases} (E + L\bar{C})\hat{\bar{x}}(i+1, j+1) = (\bar{A}_1 - K_1\bar{C})\hat{\bar{x}}(i, j+1) + (\bar{A}_2 - K_2\bar{C})\hat{\bar{x}}(i+1, j) + \bar{B}_1 u(i, j+1) + \bar{B}_2 u(i+1, j) + K_1 y(i, j+1) \\ \qquad + K_2 y(i+1, j) + Ly(i+1, j+1) + \bar{M}_1 \hat{f}(i, j+1) + \bar{M}_2 \hat{f}(i+1, j) \\ \hat{f}(i+1, j+1) = A_{d1}\hat{f}(i, j+1) + A_{d2}\hat{f}(i+1, j) + K_{d1}\bar{C}[\bar{x}(i, j+1) - \hat{\bar{x}}(i, j+1)] + K_{d2}\bar{C}[\bar{x}(i+1, j) - \hat{\bar{x}}(i+1, j)] \end{cases} \tag{17}$$

□

According to the definitions of $e(i, j)$ and $e_d(i, j)$, the error dynamic system can be presented as:

$$\begin{cases} (E + L\bar{C})e(i+1, j+1) = (\bar{A}_1 - K_1\bar{C})e(i, j+1) + (\bar{A}_2 - K_2\bar{C})e(i+1, j) \\ \qquad + \bar{M}_1 e_d(i, j+1) + \bar{M}_2 e_d(i+1, j) \\ e_d(i+1, j+1) = A_{d1}e_d(i, j+1) + A_{d2}e_d(i+1, j) \\ \qquad - K_{d1}\bar{C}e(i, j+1) - K_{d2}\bar{C}e(i+1, j) \end{cases} \tag{18}$$

Rewrite (18) into a compact form:

$$\begin{aligned} \underbrace{\begin{bmatrix} E + L\bar{C} & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}}_{\tilde{E}_L} \underbrace{\begin{bmatrix} e(i+1, j+1) \\ e_d(i+1, j+1) \end{bmatrix}}_{\tilde{e}(i+1,j+1)} &= \left(\underbrace{\begin{bmatrix} \bar{A}_1 & \bar{M}_1 \\ 0 & A_{d1} \end{bmatrix}}_{\tilde{A}_1} - \underbrace{\begin{bmatrix} K_1 \\ K_{d1} \end{bmatrix}}_{\tilde{K}_1} \underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}} \right) \underbrace{\begin{bmatrix} e(i, j+1) \\ e_d(i, j+1) \end{bmatrix}}_{\tilde{e}(i,j+1)} \\ &+ \left(\underbrace{\begin{bmatrix} \bar{A}_2 & \bar{M}_2 \\ 0 & A_{d2} \end{bmatrix}}_{\tilde{A}_2} - \underbrace{\begin{bmatrix} K_2 \\ K_{d2} \end{bmatrix}}_{\tilde{K}_2} \underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}} \right) \underbrace{\begin{bmatrix} e(i+1, j) \\ e_d(i+1, j) \end{bmatrix}}_{\tilde{e}(i+1,j)} \end{aligned} \tag{19}$$

Furthermore, it can be rewritten as:

$$\tilde{e}(i+1, j+1) = (\tilde{E}_L)^{-1}(\tilde{A}_1 - \tilde{K}_1\tilde{C})\tilde{e}(i, j+1) + (\tilde{E}_L)^{-1}(\tilde{A}_2 - \tilde{K}_2\tilde{C})\tilde{e}(i+1, j) \tag{20}$$

Referring to Lemma 1, if

$$rank(I_{n+p+q} - z_1\tilde{\tilde{A}}_1 - z_2\tilde{\tilde{A}}_2) = n + p + q. \quad \forall (z_1, z_2) \in \bar{U}^2 \tag{21}$$

holds, where $\tilde{\tilde{A}}_1 = (\tilde{E}_L)^{-1}(\tilde{A}_1 - \tilde{K}_1\tilde{C})$, $\tilde{\tilde{A}}_2 = (\tilde{E}_L)^{-1}(\tilde{A}_2 - \tilde{K}_2\tilde{C})$, then the error dynamic system (20) is asymptotically stable and $\lim_{i,j\to\infty} \tilde{e}(i, j) = \mathbf{0}$. Since the left side of (21) is related to both system and observer gain matrices, further analysis of the existence condition is essential. Partitioning $K_1 = \left[K_{11}^T \ K_{12}^T \right]^T, K_2 = \left[K_{21}^T \ K_{22}^T \right]^T, K_{d1} = \left[K_{d11}^T \ K_{d12}^T \right]^T, K_{d2} = \left[K_{d21}^T \ K_{d22}^T \right]^T$, (21) becomes:

$$
\begin{aligned}
& rank \left[I_{n+p+q} - (\tilde{E}_L)^{-1}(\tilde{A}_1 - \tilde{K}_1\tilde{C})z_1 - (\tilde{E}_L)^{-1}(\tilde{A}_2 - \tilde{K}_2\tilde{C})z_2 \right] \\
& = rank \left[\tilde{E}_L - (\tilde{A}_1 - \tilde{K}_1\tilde{C})z_1 - (\tilde{A}_2 - \tilde{K}_2\tilde{C})z_2 \right] \\
& = rank \begin{bmatrix} E + L\bar{C} - \bar{A}_1 z_1 - \bar{A}_2 z_2 + K_1\bar{C}z_1 + K_2\bar{C}z_2 & -\bar{M}_1 z_1 - \bar{M}_2 z_2 \\ K_{d1}\bar{C}z_1 + K_{d2}\bar{C}z_2 & I_q - A_{d1}z_1 - A_{d2}z_2 \end{bmatrix} \\
& = rank \begin{bmatrix} I_n - A_1 z_1 - A_2 z_2 & L_1 + K_{11}z_1 + K_{21}z_2 & -M_1 z_1 - M_2 z_2 \\ \mathbf{0} & L_2 + K_{12}z_1 + K_{22}z_2 & \mathbf{0} \\ \mathbf{0} & K_{d1}z_1 + K_{d2}z_2 & I_q - A_{d1}z_1 - A_{d2}z_2 \end{bmatrix} \begin{bmatrix} I_n & \mathbf{0} & \mathbf{0} \\ C & I_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_q \end{bmatrix} \\
& = rank \begin{bmatrix} I_n - A_1 z_1 - A_2 z_2 & -M_1 z_1 - M_2 z_2 & L_1 + K_{11}z_1 + K_{21}z_2 \\ \mathbf{0} & I_q - A_{d1}z_1 - A_{d2}z_2 & K_{d1}z_1 + K_{d2}z_2 \\ \mathbf{0} & \mathbf{0} & L_2 + K_{12}z_1 + K_{22}z_2 \end{bmatrix}
\end{aligned} \tag{22}
$$

Hence, the necessary and sufficient conditions that the error dynamic system (20) is asymptotically stable can be given as follows:

$$
rank \begin{bmatrix} I_n - A_1 z_1 - A_2 z_2 & -M_1 z_1 - M_2 z_2 & L_1 + K_{11}z_1 + K_{21}z_2 \\ \mathbf{0} & I_q - A_{d1}z_1 - A_{d2}z_2 & K_{d1}z_1 + K_{d2}z_2 \\ \mathbf{0} & \mathbf{0} & L_2 + K_{12}z_1 + K_{22}z_2 \end{bmatrix} = n + p + q \tag{23}
$$

Equation (23) is equivalent to:

$$
\begin{cases} rank(I_n - z_1 A_1 - z_2 A_2) = n \\ rank(I_q - z_1 A_{d1} - z_2 A_{d2}) = q \end{cases} \quad \forall (z_1, z_2) \in \bar{U}^2
$$

and

$$
rank\,(L_2 + K_{12}z_1 + K_{22}z_2) = p, \quad \forall\,(z_1, z_2) \in \bar{U}^2 \tag{24}
$$

Evidently, (11) is a necessary condition for Eq. (23). From another perspective, if (11) holds, L_2, K_{12} and K_{22} exist such that (23) holds. For example, the simplest choice is that $K_{12} = K_{22} = 0$ and L_2 is nonsingular. Therefore, (11) is also a sufficient condition for (23). Referring to Definition 2, it completes the proof of Theorem 1.

Remark 2 If we augment both system state and faults into one vector as $\left[x\,(i, j) \ f_s\,(i, j) \ f\,(i, j) \right]$, the 2-D descriptor system will become as follows:

$$
\begin{aligned}
\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_q \end{bmatrix} \begin{bmatrix} x(i+1, j+1) \\ f_s(i+1, j+1) \\ f(i+1, j+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 & M_1 \\ 0 & 0 & 0 \\ 0 & 0 & A_{d1} \end{bmatrix} \begin{bmatrix} x(i, j+1) \\ f_s(i, j+1) \\ f(i, j+1) \end{bmatrix} + \begin{bmatrix} A_2 & 0 & M_2 \\ 0 & 0 & 0 \\ 0 & 0 & A_{d2} \end{bmatrix} \begin{bmatrix} x(i+1, j) \\ f_s(i+1, j) \\ f(i+1, j) \end{bmatrix} \\
&\quad + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} u(i, j+1) + \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix} u(i+1, j) \\
y(i, j) &= \begin{bmatrix} C & I_p & 0 \end{bmatrix} \begin{bmatrix} x(i, j) \\ f_s(i, j) \\ f(i, j) \end{bmatrix}
\end{aligned}
$$

Notably

$$(E+LC)_1=\begin{bmatrix} I_n & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_q\end{bmatrix}+\begin{bmatrix} L_1\\ L_2\\ L_3\end{bmatrix}\begin{bmatrix} C & I_p & 0\end{bmatrix}=\begin{bmatrix} I_n+L_1C & L_1 & 0\\ L_2C & L_2 & 0\\ L_3C & L_3 & I_q\end{bmatrix}$$

In Theorem 1,

$$(E+L\bar{C})_2=\begin{bmatrix} I_n & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix} L_1\\ L_2\end{bmatrix}\begin{bmatrix} C & I_p\end{bmatrix}=\begin{bmatrix} I_n+L_1C & L_1\\ L_2C & L_2\end{bmatrix}$$

During the procedure of observer design, one needs to calculate $(E+LC)^{-1}$. In this case, from mathematical perspective, $(E+LC)_1^{-1}$ is more difficult to calculate than $(E+LC)_2^{-1}$.

Remark 3 For 1-D systems, like (Gao and Wang 2006), the necessary and sufficient conditions for an asymptotically stable observer are commonly obtained by investigating the detectability of the system. However, the detectability is only a necessary condition for a 2-D asymptotically stable observer, as pointed out in (Bisiacco 1986) three decades ago. Hence, analyzing the detectability of the 2-D system is no longer applicable, which is different from the 1-D case.

If system (1) and (3) is not open-loop asymptotically stable, and rank condition (11) is not satisfied, the asymptotically stable observer will not exist. To deal with this problem, another kind of observer, the uniformly ultimately bounded observer, is proposed.

3.2 Uniformly ultimately bounded observer

Define the following matrices:

$$E=\begin{bmatrix} I_n & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\quad \bar{A}_{1new}=\begin{bmatrix} A_1 & \mathbf{0}\\ \mathbf{0} & -I_P\end{bmatrix},\quad \bar{A}_{2new}=\begin{bmatrix} A_2 & \mathbf{0}\\ \mathbf{0} & -I_P\end{bmatrix},\quad \bar{B}_1=\begin{bmatrix} B_1\\ \mathbf{0}\end{bmatrix},\quad \bar{B}_2=\begin{bmatrix} B_2\\ \mathbf{0}\end{bmatrix},$$
$$N=\begin{bmatrix} \mathbf{0}\\ I_P\end{bmatrix},\quad \bar{C}=\begin{bmatrix} C & I_p\end{bmatrix},\quad \bar{M}_1=\begin{bmatrix} M_1\\ \mathbf{0}\end{bmatrix},\quad \bar{M}_2=\begin{bmatrix} M_2\\ \mathbf{0}\end{bmatrix} \tag{25}$$

One can obtain the following augmented 2-D singular systems:

$$\begin{cases} E\bar{x}(i+1,j+1)=\bar{A}_{1new}\bar{x}(i,j+1)+\bar{A}_{2new}\bar{x}(i+1,j)+\bar{B}_1u(i,j+1)+\bar{B}_2u(i+1,j)\\ \qquad\qquad +\bar{M}_1f(i,j+1)+\bar{M}_2f(i+1,j)+N_1f_s(i,j+1)+N_2f_s(i+1,j)\\ y(i,j)=\bar{C}\bar{x}(i,j)\\ f(i+1,j+1)=A_{d1}f(i,j+1)+A_{d2}f(i+1,j)\end{cases} \tag{26}$$

and unknown boundary conditions satisfy:

$$\begin{aligned} &\sup\|\bar{x}(i,0)\|<\infty,\quad i=0,1,\ldots;\quad \sup\|\bar{x}(0,j)\|<\infty,\quad j=1,2,\ldots\\ &\sup\|f(i,0)\|<\infty,\quad i=0,1,\ldots;\quad \sup\|f(0,j)\|<\infty,\quad j=1,2,\ldots\end{aligned} \tag{27}$$

we assume $v=[\,v_1\;v_2\;\ldots\;v_p\,]^T$ is the upper bound of f_s.

Theorem 2 *For system (26) with boundary conditions (27), if and only if:*

$$rank\begin{bmatrix} I_n-A_1z_1-A_2z_2 & L_1+K_{11}z_1+K_{21}z_2 & -M_1z_1-M_2z_2\\ -C\,(z_1+z_2) & L_2+K_{12}z_1+K_{22}z_2+I_p\,(z_1+z_2) & \mathbf{0}\\ \mathbf{0} & K_{d1}z_1+K_{d2}z_2 & I_q-A_{d1}z_1-A_{d2}z_2\end{bmatrix}$$
$$=n+p+q,\quad \forall(z_1,z_2)\in\bar{U}^2. \tag{28}$$

holds, then there exist the following observer matrices:

$$
\begin{aligned}
&L, K_1, K_2 \in R^{(n+p)\times p}, K_{d1}, K_{d2} \in R^{q\times p},\\
&F_1 = (E+L\bar{C})^{-1}(\bar{A}_{1new} - K_1\bar{C}), F_2 = (E+L\bar{C})^{-1}(\bar{A}_{2new} - K_2\bar{C}),\\
&H_1 = (E+L\bar{C})^{-1}\bar{B}_1, H_2 = (E+L\bar{C})^{-1}\bar{B}_2, T = (E+L\bar{C})^{-1}L,\\
&R_1 = (E+L\bar{C})^{-1}\bar{M}_1, R_2 = (E+L\bar{C})^{-1}\bar{M}_2,\\
&G_1 = (E+L\bar{C})^{-1}\bar{A}_1(E+L\bar{C})^{-1}L, G_2 = (E+L\bar{C})^{-1}\bar{A}_2(E+L\bar{C})^{-1}L
\end{aligned} \tag{29}
$$

and boundary conditions (13), *which can lead to* (4) *is a uniformly ultimately bounded observer for system* (26). *The uniformly ultimately bounded error is given as follows*:

$$
\left\| \begin{matrix} e(i+1, j+1) \\ e_d(i+1, j+1) \end{matrix} \right\| \leq \zeta, \zeta > 0, \quad \forall i \geq I, \forall j \geq J \tag{30}
$$

Proof We divide the proof into two steps. □

Step 1

For Eq. (26), by adding $Ly(i+1, j+1)$ to both sides, one can obtain:

$$
\begin{cases}
(E+L\bar{C})\bar{x}(i+1, j+1) = (\bar{A}_{1new} - K_1\bar{C})\bar{x}(i, j+1) + (\bar{A}_{2new} - K_2\bar{C})\bar{x}(i+1, j)\\
\qquad + \bar{B}_1u(i, j+1) + \bar{B}_2u(i+1, j) + \bar{M}_1 f(i, j+1)\\
\qquad + \bar{M}_2 f(i+1, j) + Nf_s(i, j+1) + Nf_s(i+1, j) + Ly(i+1, j+1)\\
\qquad + K_1y(i, j+1) + K_2y(i+1, j)\\
f(i+1, j+1) = A_{d1}f(i, j+1) + A_{d2}f(i+1, j)
\end{cases} \tag{31}
$$

Substitute $z(i, j) = \hat{\bar{x}}(i, j) - (E+L\bar{C})^{-1}Ly(i, j)$ into (4) and use (12), the observer in (12) becomes:

$$
\begin{cases}
(E+L\bar{C})\hat{\bar{x}}(i+1, j+1) = (\bar{A}_{1new} - K_1\bar{C})\hat{\bar{x}}(i, j+1) + (\bar{A}_{2new} - K_2\bar{C})\hat{\bar{x}}(i+1, j)\\
\qquad + \bar{B}_1u(i, j+1) + \bar{B}_2u(i+1, j) + K_1y(i, j+1)\\
\qquad + K_2y(i+1, j) + Ly(i+1, j+1)\\
\qquad + \bar{M}_1\hat{f}(i, j+1) + \bar{M}_2\hat{f}(i+1, j)\\
\hat{f}(i+1, j+1) = A_{d1}\hat{f}(i, j+1) + A_{d2}\hat{f}(i+1, j) + K_{d1}\bar{C}[\bar{x}(i, j+1) - \hat{\bar{x}}(i, j+1)]\\
\qquad + K_{d2}\bar{C}[\bar{x}(i+1, j) - \hat{\bar{x}}(i+1, j)]
\end{cases} \tag{32}
$$

Therefore, the error dynamic system can be obtained as follows:

$$
\begin{cases}
(E+L\bar{C})e(i+1, j+1) = (\bar{A}_{1new} - K_1\bar{C})e(i, j+1) + (\bar{A}_{2new} - K_2\bar{C})e(i+1, j)\\
\qquad + Nf_s(i, j+1) + Nf_s(i+1, j) + \bar{M}_1e_d(i, j+1) + \bar{M}_2e_d(i+1, j)\\
e_d(i+1, j+1) = A_{d1}e_d(i, j+1) + A_{d2}e_d(i+1, j) - K_d\bar{C}[e(i, j+1) + e(i+1, j)]
\end{cases} \tag{33}
$$

Error system (33) can be presented in a compact form as follows:

$$\underbrace{\begin{bmatrix} E+L\bar{C} & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}}_{\tilde{E}_L} \underbrace{\begin{bmatrix} e(i+1,j+1) \\ e_d(i+1,j+1) \end{bmatrix}}_{\tilde{e}(i+1,j+1)} = \left(\underbrace{\begin{bmatrix} \bar{A}_{1new} & \bar{M}_1 \\ \mathbf{0} & A_{d1} \end{bmatrix}}_{\tilde{A}_{1new}} - \underbrace{\begin{bmatrix} K_1 \\ K_{d1} \end{bmatrix}}_{\tilde{K}_1} \underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}} \right) \begin{bmatrix} e(i,j+1) \\ e_d(i,j+1) \end{bmatrix}$$
$$+ \left(\underbrace{\begin{bmatrix} \bar{A}_{2new} & \bar{M}_2 \\ \mathbf{0} & A_{d2} \end{bmatrix}}_{\tilde{A}_{2new}} - \underbrace{\begin{bmatrix} K_2 \\ K_{d2} \end{bmatrix}}_{\tilde{K}_2} \underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}} \right) \begin{bmatrix} e(i+1,j) \\ e_d(i+1,j) \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} N \\ \mathbf{0} \end{bmatrix}}_{\tilde{N}} f_s(i,j+1) + \underbrace{\begin{bmatrix} N \\ \mathbf{0} \end{bmatrix}}_{\tilde{N}} f_s(i+1,j) \tag{34}$$

Furthermore, (34) can be rewritten as:

$$\begin{aligned} \tilde{e}(i+1,j+1) =& (\tilde{E}_L)^{-1}(\tilde{A}_{1new} - \tilde{K}_1\tilde{C})\tilde{e}(i,j+1) \\ &+ (\tilde{E}_L)^{-1}(\tilde{A}_{2new} - \tilde{K}_2\tilde{C})\tilde{e}(i+1,j) \\ &+ (\tilde{E}_L)^{-1}\tilde{N} f_s(i,j+1) + (\tilde{E}_L)^{-1}\tilde{N} f_s(i+1,j) \\ =& \tilde{A}_{e1n}\tilde{e}(i,j+1) + \tilde{A}_{e2n}\tilde{e}(i+1,j) \\ &+ \tilde{F}_{e1} f_s(i,j+1) + \tilde{F}_{e2} f_s(i+1,j) \end{aligned} \tag{35}$$

To obtain the uniformly ultimately bounded estimation of system (26), the open-loop error system (35) should be asymptotically stable. Referring to Lemma 1, if and only if the following rank condition holds:

$$rank(I_{n+p+q} - z_1\tilde{A}_{e1n} - z_2\tilde{A}_{e2n}) = n+p+q. \quad \forall (z_1, z_2) \in \bar{U}^2 \tag{36}$$

It becomes:

$$\begin{aligned} & rank\left[I_{n+p+q} - (\tilde{E}_L)^{-1}(\tilde{A}_{1new} - \tilde{K}_1\tilde{C})z_1 - (\tilde{E}_L)^{-1}(\tilde{A}_{2new} - \tilde{K}_2\tilde{C})z_2 \right] \\ & = rank \begin{bmatrix} I_n - A_1z_1 - A_2z_2 & L_1 + K_{11}z_1 + K_{21}z_2 & -M_1z_1 - M_2z_2 \\ -C\,(z_1+z_2) & L_2 + K_{12}z_1 + K_{22}z_2 + I_p\,(z_1+z_2) & \mathbf{0} \\ \mathbf{0} & K_{d1}z_1 + K_{d2}z_2 & I_q - A_{d1}z_1 - A_{d2}z_2 \end{bmatrix} \end{aligned} \tag{37}$$

Thus, the necessary and sufficient conditions for the asymptotical stability of (35) can be given as follows:

$$\begin{aligned} & rank \begin{bmatrix} I_n - A_1z_1 - A_2z_2 & L_1 + K_{11}z_1 + K_{21}z_2 & -M_1z_1 - M_2z_2 \\ -C\,(z_1+z_2) & L_2 + K_{12}z_1 + K_{22}z_2 + I_p\,(z_1+z_2) & \mathbf{0} \\ \mathbf{0} & K_{d1}z_1 + K_{d2}z_2 & I_q - A_{d1}z_1 - A_{d2}z_2 \end{bmatrix} \\ & = n+p+q \end{aligned}$$

Step 2

Given that (35) is open-loop asymptotically stable, the effect of the bounded boundary conditions of the error dynamic system will become zero when $i \to \infty$ and $j \to \infty$. In this case, only the effects of the measurement fault on the estimation error should be further analyzed to obtain the final value of the error vector.

According to Bose (1982) and referring to Definition 1, the transfer function from f_s to $\tilde{e}$ is:

$$E_e(z_1, z_2) = [I_{n+p+q} - z_1\tilde{A}_{e1n} - z_2\tilde{A}_{e2n}]^{-1}[z_1\tilde{F}_{e1} + z_2\tilde{F}_{e2}]F_s(z_1, z_2) \tag{38}$$

In this case, $E_e(z_1, z_2)$ and $F_s(z_1, z_2)$ are the Z transformation of $\tilde{e}(i, j)$ and $f_s(i, j)$, respectively. Using the 2-D Z transformation formula (6) and the 2-D final-value theorem (7), the final value of the estimation error is calculated as follows:

$$\begin{aligned}
\lim_{i\to\infty, j\to\infty} |\tilde{e}(i, j)| &= \lim_{z_1\to 1, z_2\to 1} \left| \frac{(1-z_1)(1-z_2)}{z_1z_2} E_e(z_1, z_2) \right| \\
&= \lim_{z_1\to 1, z_2\to 1} \left| \frac{(1-z_1)(1-z_2)}{z_1z_2} [I_{n+p+q} - z_1\tilde{A}_{e1n} - z_2\tilde{A}_{e2n}]^{-1}[z_1\tilde{F}_{e1} + z_2\tilde{F}_{e2}]F_s(z_1, z_2) \right| \\
&\le \left| [I_{n+p+q} - \tilde{A}_{e1n} - \tilde{A}_{e2n}]^{-1}[\tilde{F}_{e1} + \tilde{F}_{e2}] \right| v \\
&= \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_{n+p+q} \end{bmatrix}^T
\end{aligned} \tag{39}$$

By defining $\zeta/2 = \kappa = \left\| \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_{n+p+q} \end{bmatrix}^T \right\|$, I and J exist such that $\|\tilde{e}(i, j)\| \le 2\kappa = \zeta$, $\forall i \ge I$,$\forall j \ge J$. If $f_s(i, j)$ is bounded, the gains K_1, K_2,K_{d1},K_{d2} and L can be found to make the observer in Theorem 2 an uniform ultimately bounded observer with the bound value ζ. Referring to Definition 3, the proof of Theorem 2 is completed.

Remark 4 Notice that

$$[\tilde{F}_{e1} + \tilde{F}_{e2}] = 2\begin{bmatrix} I_n & -L_1(L_2)^{-1} & \mathbf{0} \\ -C & (I_p + CL_1)(L_2)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_q \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ I_P \\ \mathbf{0} \end{bmatrix} = 2\begin{bmatrix} -L_1(L_2)^{-1} \\ (I_p + CL_1)(L_2)^{-1} \\ \mathbf{0} \end{bmatrix} \tag{40}$$

In this case, if $L_2^{-1} \to \mathbf{0}$, $[\tilde{F}_{e1} + \tilde{F}_{e2}] \to \mathbf{0}$, then $\zeta \to 0$. Thus, the final error upper bound of the observer in Theorem 2 can be designed to be arbitrarily small by appropriately choosing L.

Remark 5 For the designed observers, one needs to choose appropriate L, $\tilde{K}_1$ and $\tilde{K}_2$ to satisfy (21) and (36). According to Theorem 4 in Xu et al. (2012) and Theorem 1 in Hinamoto (1993), after choosing L, the design of $\tilde{K}_1$ and $\tilde{K}_2$ can be formulated into a LMI problem, and the LMI problem can be conveniently solved by using software tools such as the MATLAB LMI tool box.

The rank condition (28) is not only related to the observer gains, but also related to the system matrices; it is not easy to check (28) before the observer design. Therefore, only the rank condition related to the system matrices is required.

Theorem 3 *For the system* (26) *with boundary conditions given by* (27), (41) *is a necessary condition for the existence of uniform ultimately bounded observer.*

$$\left\{ \begin{array}{l} rank\begin{bmatrix} I_n - A_1z_1 - A_2z_2 \\ C \end{bmatrix} = n, rank\left(I_q - A_{d1}z_1 - A_{d2}z_2\right) = q. \\ when\ (z_1 + z_2) \ne 0 \\ rank\,[I_n - A_1z_1 - A_2z_2] = n, rank\left(I_q - A_{d1}z_1 - A_{d2}z_2\right) = q. \\ when\ (z_1 + z_2) = 0 \end{array} \right\}, \quad \forall (z_1, z_2) \in \bar{U}^2 \tag{41}$$

Proof To eliminate the effects of the observer matrices, one needs to perform simple matrix decomposition for (36), which can be obtained as:

$$\begin{aligned} & rank(I_{n+p+q} - z_1\tilde{A}_{e1n} - z_2\tilde{A}_{e2n}) \\ & = rank\left\{\begin{bmatrix} I_{n+p+q} & (\tilde{E}_L)^{-1}K_1 z_1 & (\tilde{E}_L)^{-1}K_2 z_2 \end{bmatrix}\begin{bmatrix} I_{n+p+q} - (\tilde{E}_L)^{-1}\tilde{A}_1 - (\tilde{E}_L)^{-1}\tilde{A}_2 \\ \tilde{C} \\ \tilde{C} \end{bmatrix}\right\} \end{aligned} \tag{42}$$

The necessary condition of $rank(I_{n+p+q} - z_1\tilde{A}_{e1n} - z_2\tilde{A}_{e2n}) = n + p + q$ is

$$rank\begin{bmatrix} I_{n+p+q} - (\tilde{E}_L)^{-1}\tilde{A}_1 - (\tilde{E}_L)^{-1}\tilde{A}_2 \\ \tilde{C} \\ \tilde{C} \end{bmatrix} = n + p + q$$

Considering that

$$\begin{aligned} & rank\begin{bmatrix} I_{n+p+q} - (\tilde{E}_L)^{-1}\tilde{A}_1 - (\tilde{E}_L)^{-1}\tilde{A}_2 \\ \tilde{C} \\ \tilde{C} \end{bmatrix} \\ & = rank\begin{bmatrix} I_{n+p+q} - (\tilde{E}_L)^{-1}\tilde{A}_1 - (\tilde{E}_L)^{-1}\tilde{A}_2 \\ \tilde{C} \end{bmatrix} \end{aligned}$$

Therefore, the necessary condition of $rank(I_{n+p+q} - z_1\tilde{A}_{e1n} - z_2\tilde{A}_{e2n}) = n + p + q$ equals

$$rank\begin{bmatrix} I_{n+p+q} - (\tilde{E}_L)^{-1}\tilde{A}_1 - (\tilde{E}_L)^{-1}\tilde{A}_2 \\ \tilde{C} \end{bmatrix} = n + p + q \tag{43}$$

One can obtain that

$$\begin{aligned} & rank\begin{bmatrix} I_{n+p+q} - (\tilde{E}_L)^{-1}\tilde{A}_1 - (\tilde{E}_L)^{-1}\tilde{A}_2 \\ \tilde{C} \end{bmatrix} \\ & = rank\begin{bmatrix} (\tilde{E}_L)^{-1} & \mathbf{0} \\ \mathbf{0} & I_p \end{bmatrix}\begin{bmatrix} \tilde{E}_L - \tilde{A}_1 z_1 - \tilde{A}_2 z_2 \\ \tilde{C} \end{bmatrix} \\ & = rank\begin{bmatrix} \tilde{E}_L - \tilde{A}_1 z_1 - \tilde{A}_2 z_2 \\ \tilde{C} \end{bmatrix} \\ & = rank\begin{bmatrix} I_n - A_1 z_1 - A_2 z_2 & \mathbf{0} & -M_1 z_1 - M_2 z_2 \\ \mathbf{0} & I_p(z_1 + z_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_q - A_{d1} z_1 - A_{d2} z_2 \\ C & I_p & \mathbf{0} \end{bmatrix} \\ & = \left\{\begin{matrix} rank\begin{bmatrix} I_n - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix} + p + rank\left(I_q - A_{d1} z_1 - A_{d2} z_2\right), \text{ when } (z_1 + z_2) \neq 0 \\ rank\left[I_n - A_1 z_1 - A_2 z_2\right] + p + rank\left(I_q - A_{d1} z_1 - A_{d2} z_2\right), \text{ when } (z_1 + z_2) = 0 \end{matrix}\right\} \end{aligned} \tag{44}$$

holds, so (41) is a necessary condition for (36). The proof of the final bounded estimation error is the same as that of Theorem 2. Therefore, the proof of Theorem 3 is completed. □

Remark 6 Summarizing the relationship of three theorems, if the systems satisfy the rank conditions given in Theorems 1 and 2, there will exist observers that can simultaneously estimate the states and faults, and constructive design methods are proposed. If the systems do not satisfy these rank conditions, then there is no observer that can estimate the states and faults. However, in Theorem 2, the rank conditions are not only related to the system matrices, but also related to the observer gains, so the rank conditions are difficult to assess before the observer design. In Theorem 3, only the rank conditions that are related to the system matrices are proposed.

Remark 7 This study indicates that the uniformly ultimately bounded observers for 2-D systems with multiple faults have dramatically different existence conditions, compared with their 1-D counterparts. For the uniformly ultimately bounded observer, the results for 2-D systems presented in Theorems 2 and 3 are more complex than those for 1-D systems. Furthermore, the upper bound of the uniformly ultimately bounded observer is presented and analyzed, which is much more complicated than the 1-D case.

4 Simulation examples

In this section, two examples are provided to demonstrate the feasibility and effectiveness of the proposed observer design methods in the previous theorems.

4.1 Example 1

Consider the thermal process described by the following partial differential equation (Kaczorek 1985)

$$\begin{aligned} T_x(x,t) &= -T_t(x,t) - T(x,t) + bu(x,t) \\ T(0,t) &= l_1(t), T(x,0) = l_2(x), x \in [0, x_L], t \in [0, \infty] \end{aligned} \tag{45}$$

where $T(x,t)$ is the temperature at space point x and time point t, $u(x,t)$ is the given force function, $T(0,t)$ and $T(x,0)$ are the boundary and initial conditions, respectively, $l_1(t)$ and $l_2(x)$ are known functions. Equation (45) is usually used to describe the thermal process such as heat exchangers, chemical reactors, and pipe furnaces (Kaczorek 1985). Discretize (45) and define

$$\begin{aligned} T(i,j) &= T(i\Delta x, j\Delta t) \\ T_x\,(x,t) &= \frac{T(i,j) - T(i-1,j)}{\Delta x} \\ T_t\,(x,t) &= \frac{T(i,j+1) - T(i,j)}{\Delta t} \\ x^T(i,j) &= \left[\, T^T(i-1,j)\ T^T(i,j) \,\right]^T \end{aligned} \tag{46}$$

where Δx and Δt are the space step and time step, respectively. Let $\Delta t = 0.1, \Delta x = 0.4, b = 0$. By substituting (46) into (45), we can easily yield a 2-D FM-II state space model with

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - \Delta t \end{bmatrix} \\ M_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 100 \end{bmatrix}^T \end{aligned} \tag{47}$$

and the dynamics of the faults can be assumed as follows:

$$\begin{aligned} & f(i,j) = A_{d1} f(i-1,j) + A_{d2} f\,(i,j-1)\,, A_{d1} = 0.5, A_{d2}{=}0.4 \\ & f_s(i,j) = 20\sin(i+j) \end{aligned} \tag{48}$$

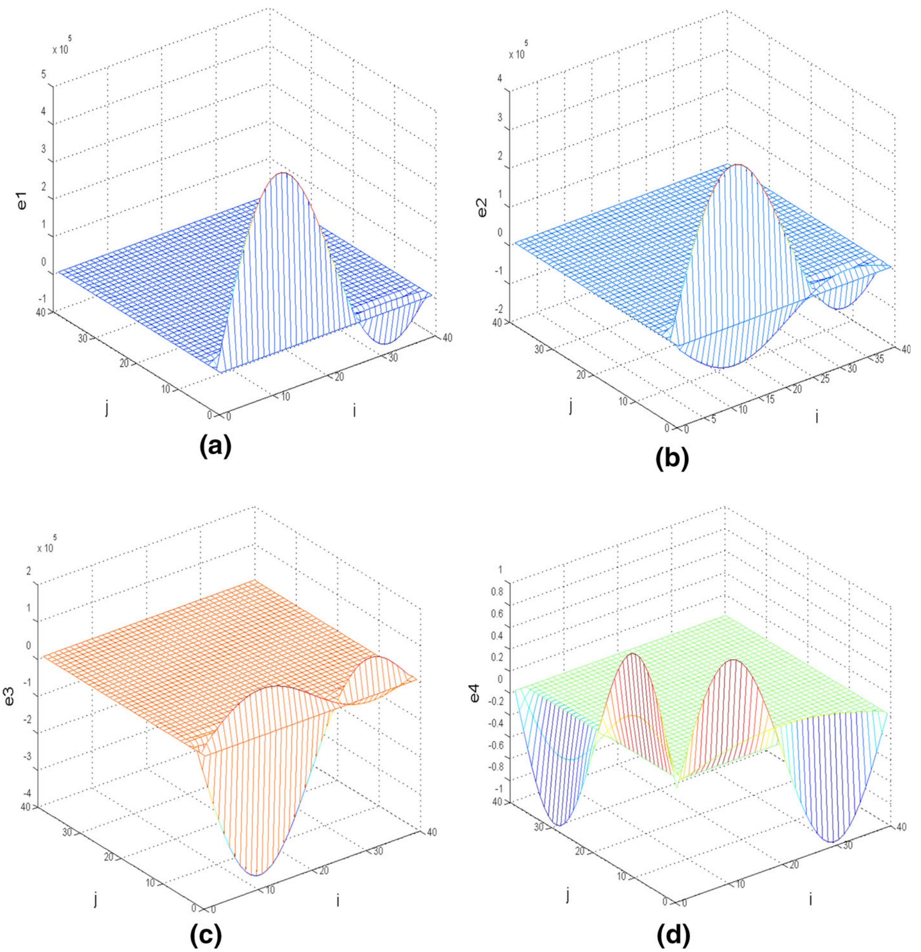

Fig. 1 States and faults estimation errors of the 2-D system presented in (47) are shown in subplots (**a**), (**b**), (**c**), and (**d**), respectively

The boundary conditions of the observer in this example are assumed to be zero, so the boundary conditions of the error dynamic system equal the boundary conditions of the original 2-D system. The boundary conditions are similar to those in example 2. In the rectangular region, the boundary conditions of the error are:

$$\begin{aligned}
&e_1(0, j) = \sin(2\pi j/l), e_1(i, 0) = \sin(\pi j/l),\\
&e_2(0, j) = \sin(2\pi j/l), e_2(i, 0) = \sin(2\pi j/l),\\
&e_3(0, j) = \sin(2\pi j/l), e_3(i, 0) = \sin(\pi j/l),\\
&e_4(0, j) = \sin(2\pi j/l), e_4(i, 0) = \sin(2\pi j/l), 0 \le i, j \le l = 40
\end{aligned} \tag{49}$$

One can find that

$$\begin{aligned}
rank\,[I_2 - z_1 A_1 - z_2 A_2] = 2, \quad \forall\,(z_1, z_2) \in \bar{U}^2\\
rank\,[I_1 - z_1 A_{d1} - z_2 A_{d2}] = 1, \quad \forall\,(z_1, z_2) \in \bar{U}^2
\end{aligned}$$

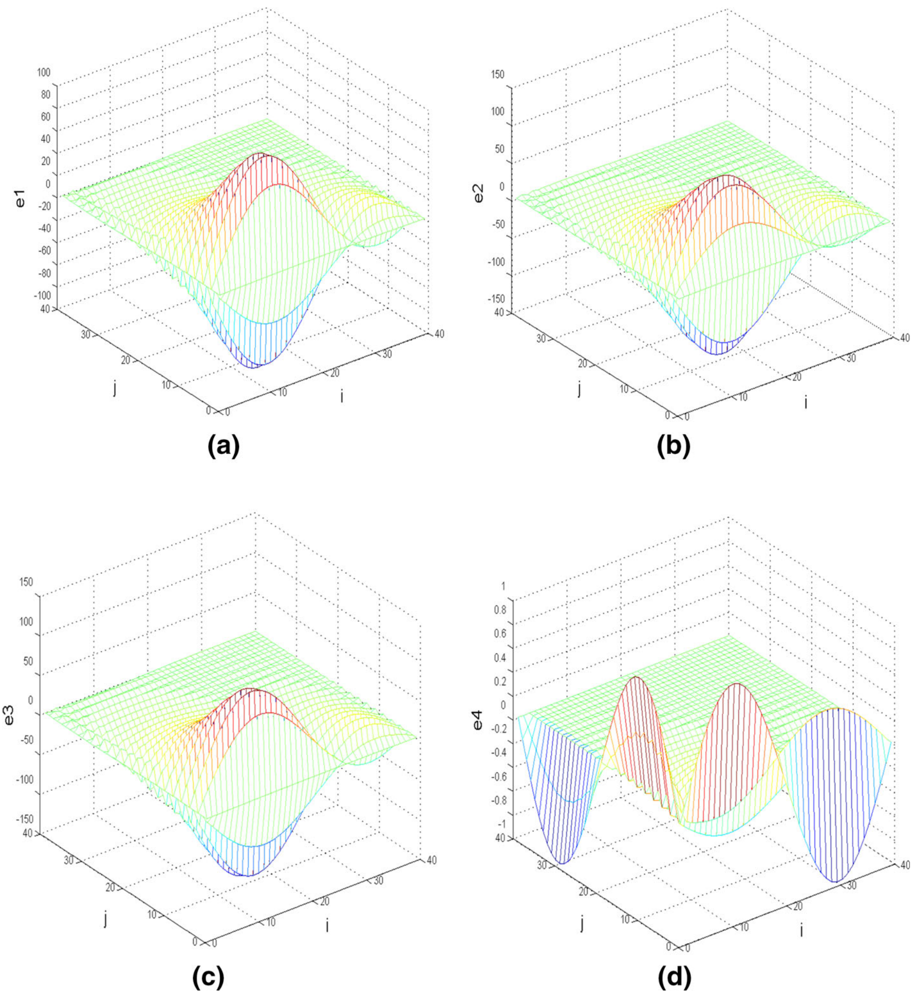

Fig. 2 States and faults estimation errors of the 2-D system presented in (51) are shown in (**a**), (**b**), (**c**), and (**d**), respectively

holds. Referring to Theorem 1, an asymptotically stable observer exists for (47). According to Theorem 4 in Xu et al. (2012) and Theorem 1 in Hinamoto (1993), one obtains the following matrices:

$$\tilde{K}_1 = \begin{bmatrix} 0.8 \\ -0.4 \\ -0.7 \\ 0.5 \end{bmatrix}, \quad \tilde{K}_2 = \begin{bmatrix} 0.45 \\ -0.35 \\ -0.3 \\ 0.1 \end{bmatrix} \tag{50}$$

4.2 Example 2

Consider the 2-D FM-II system, where the system coefficient matrices are given as follows:

$$A_1 = \begin{bmatrix} -0.8 & 0.1 \\ 0.5 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.8 & 0 \\ 1 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & -10 \end{bmatrix}^T \tag{51}$$

The dynamics of the faults are the same as example 1.

As

$$rank\,[I_2 - A_1 z_1 - A_2 z_2] = 1 < 2, \left(z_1 = -\frac{5}{8}, z_2 = -\frac{5}{8} \right)$$

holds, referring to Theorem 1, no asymptotically stable observer exists for (51). However,

$$rank \begin{bmatrix} I_2 - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix} = 2, \quad \text{when } (z_1 + z_2) \neq 0$$
$$rank\,[I_2 - A_1 z_1 - A_2 z_2] = 2, \quad \text{when } (z_1 + z_2) = 0$$
$$rank\,[I_1 - z_1 A_{d1} - z_2 A_{d2}] = 1, \quad \forall\,(z_1, z_2) \in \bar{U}^2$$

holds, which means the necessary condition for the existence of a uniformly ultimately bounded observer is satisfied. Similar to Example 1, one obtains the following matrices:

$$\tilde{K}_1 = \begin{bmatrix} 5600 \\ -3100 \\ -19 \\ -8 \end{bmatrix}, \quad \tilde{K}_2 = \begin{bmatrix} -270000 \\ 91600 \\ 340 \\ -0.03 \end{bmatrix} \tag{52}$$

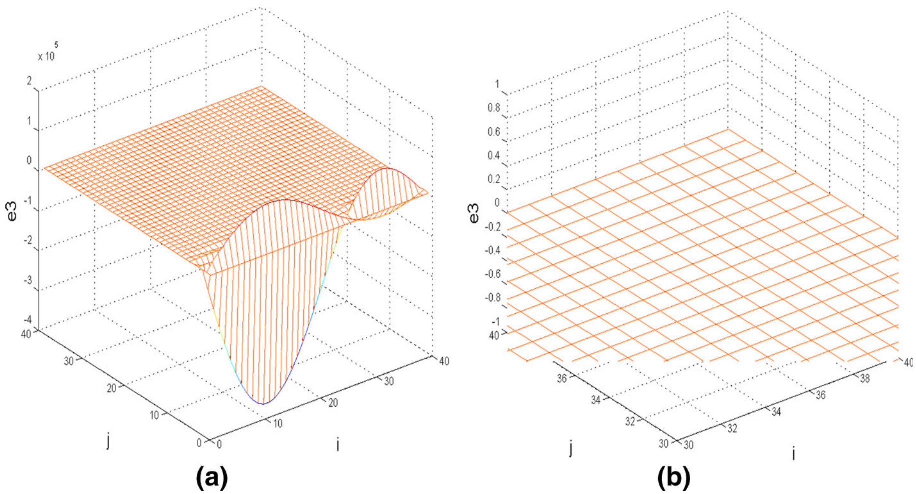

Fig. 3 Left subplot shows the fault estimation error in the measurement equation in Theorem 1 and right subplot shows the zoomed-in error

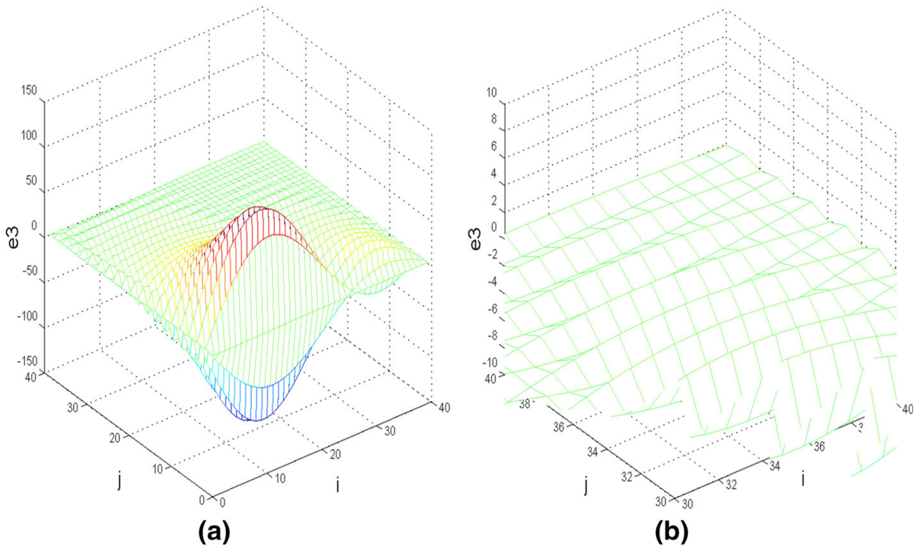

Fig. 4 Left subplot shows the fault estimation error in the measurement equation in Theorem 2 and right subplot shows the zoomed-in error

According to (19) and (34), the estimation error $e(i, j)(0 \le i, j \le 40)$ is shown in Figs. 1, 2. Both $e_1(i, j)$ and $e_2(i.j)$ represent the system states, whereas $e_3(i, j)$ and $e_4\ (i, j)$ represent the fault in the measurement equation $f_s\ (i, j)$ and fault in the states updating equation $f\ (i, j)$, respectively.

Figure 1 shows that the estimation error of both system states and two kinds of faults converge to zero rapidly. Figure 2 illustrates that the estimation error of both system states and faults converge to a small value.

To show the convergence properties of both the asymptotically stable and uniformly ultimately bounded observers clearly, we zoom in Figs. 1c and 2c to obtain magnified images of Figs. 3, 4. The fault estimation error in Fig. 1c is asymptotically stable and the fault estimation error in Fig. 2c is uniformly ultimately bound.

5 Conclusions

The existence and design of singular system observers for 2-D linear systems with multiple channel faults are investigated in this study. For different system conditions, necessary and sufficient conditions for the existence of observers are obtained. The effectiveness of the proposed methods is validated by two simulation examples. The estimation results can be used for active fault tolerant control. In the future, the estimation of 2-D nonlinear time-varying systems with multiple channel faults will be discussed.

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