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Simultaneous estimation of multiple channel faults for two-dimensional linear systems

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ABSTRACT
This study focuses on the problem of simultaneous estimation of multiple channel faults for two-dimensional linear systems, which are described by Fornasini–Marchesini second (FM-II) model, and the faults that exist in state equation and measurement equation. By transforming the fault in the measurement equation as augmented state, the FM-II model with faults in the state equation and measurement equation can be rewritten into a singular system. Hence, several observers are proposed for the singular systems, and then the estimation of the faults in the state equation and measurement equation can be obtained. Using Lyapunov stability theory, sufficient conditions for the existence of the asymptotically stable observer and uniformly ultimately bounded observer are derived in the context of time domain. For the bounded observer, the upper bound of estimation error can be provided referring to the fault bound. Numerical and practical examples are given to demonstrate the effectiveness of the proposed method.

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Two-dimensional systems; singular system; asymptotically stable observer; uniformly ultimately bounded observer; fault estimation

1. Introduction

Two-dimensional (2-D) systems theory has profound engineering background. Different state-space models for 2-D systems have been proposed to solve complex problems in different fields (Benzaouia, Hmamed, & Tadeo, 2015; Bisiacco, 1986; Cheng, Fang, & Wang, 2016; Fornasini & Marchesini, 1976, 1978; Givone & Roesser, 1973; Kaczorek, 1985; Roesser, 1975). For example, for multidimensional linear iterative circuits and image data processing, Roesser model (Givone & Roesser, 1973; Roesser, 1975) was first presented in 1973. For 2-D digital filtering and control, Fornasini–Marchesini first (FM-I) model and Fornasini–Marchesini second (FM-II) model (Fornasini & Marchesini, 1976, 1978) were first presented in 1976. In these 2-D systems, some definitions such as controllability and observability have been given in Ciftcibasi and Yuksel (1983), some control methods such as robust control and optimum control have been proposed in Ahn, Shi, and Basin (2015), De Souza and Osowsky (2013), Ghous and Xiang (2015), Liang, Wang, and Liu (2015), Lin and Bruton (1989), Xu and Yu (2009) and Zhao, Shen, and Wang (2017). Because the Roesser model and the FM-I model are special cases of the FM-II model, we focus on the 2-D systems that are described by the FM-II model.

In practice, 2-D systems face tremendous threat of faults and have the needs of reliability and security. If fault occurs in the state equation or measurement equation, the system performance will be degraded. For one-dimensional (1-D) systems, the problem of fault diagnosis and fault-tolerant control has attracted significant attention with numerous published results (Cao, Tao, Wang, Li, & Huang, 2016; Ding, 2009; Gao, 2015; Gao & Wang, 2006; Koenig, 2005; Li, Ding, Qiu, & Yang, 2017; Li, Ding, Qiu, Yang, & Zhang, 2016; Tao, Shen, Wang, & Ye, 2015; Xiao & Yin, 2016; Xiao, Yin, & Kaynak, 2016; Zhong, Song, & Ding, 2015). However, the fault diagnosis problem on 2-D systems is far from being well investigated. Note that there are a few studies about 2-D fault diagnosis that have just appeared in the literature. For linear time-invariant FM-II model with system perturbations, by accurately decoupling, Bisiacco and Valcher (2006) give the results of fault detection and isolation. Based on parity checks and polynomial matrices (Fornasini & Marchesini, 1988), the dynamical redundancy relations of 2-D systems have been obtained.

In the framework of fault diagnosis, observer/filter-based methods play a significant role in model-based fault detection and they can provide much information such as fault sizes and kinds.Therefore, there are important industrial and theoretical applications in fault diagnosis and fault-tolerant control. Some valuable results have been obtained about this topic. For example, using 2-D singular systems theory, the state and the sensor faults of FM-II model are estimated simultaneously in Zhao, Lin, and Wang (2015). In Zhao, Zhou, and Wang (2015), for 2-D nonlinear systems with time-varying delays and system perturbations, the state and sensor fault estimation is obtained simultaneously and used in fault reconstruction. In Wu, Yao, and Zheng (2012), based on generalised H_2 index, the results of fault detection for 2-D Markov jump systems are obtained under the condition of random packet dropout.

Although there are some different observer structures and different design and analysis methods for 2-D systems, the 2-D observer theory is far from satisfactory, due to the complexity of 2-D systems, especially the unique property that the 2-D systems update in two independent directions. To the best of the authors' knowledge, all reported results concentrateon the estimation of state or fault in the measurement equation. However, in practical systems, faults may not occur alone; the faults in the

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state equation and measurement equation could appear simultaneously. Therefore, simultaneous estimation of faults in the state equation and measurement equation for 2-D linear systems is significant for fault diagnosis and fault-tolerant control. Owing to the difficulty of analysis and calculation, there is no related research about the fault estimation in the state equation and it is still an open problem. This gives us the motivation for the study. Hence, the study of simultaneous estimation of faults in the measurement equation and state equation will enrich the theory of the analysis and design of 2-D observers.

Motivated by the aforementioned discussion, this paper considers the problem of estimating the faults in the state equation and measurement equation simultaneously for 2-D systems. Differently with all observers in the reported studies, new observers that include fault estimation in the state equation are first presented. By using the singular system approach and Lyapunov stability theory, we derive the sufficient conditions of the existence and calculate the new observers gain matrices. The dynamic characteristics of fault in the state equation are assumed to be known and it contains two cases. For fault in the state equation without disturbance, the sufficient condition for the existence of asymptotically stable observer is presented. For fault in the state equation with disturbance, the sufficient condition for the existence of uniformly ultimately bounded observer is obtained. By analysing whether the difference of 2-D Lyapunov function is negative, we give the upper bound of estimation error.

The main contributions of this paper can be summarised as follows. For 2-D system described by the FM-II model, when faults in the state equation and measurement equation occur simultaneously, this study first designs a new observer that can estimate the faults in the state equation and measurement equation at the same time. Furthermore, the asymptotically stable observer and uniformly ultimately bounded observer are derived.

The rest of the paper is organised as follows. In Section 2, for considered systems, some definitions and lemmas are presented. Section 3 gives the sufficient condition of asymptotically stable observer and uniformly ultimately bounded observer and derives the formulation of observers. In Section 4, the effectiveness of the proposed method is validated by numerical and practical examples. Section 5 summarises the results of this study.

Notations

R^n denotes n-dimensional space, $R^{n\times m}$ indicates the set of all $n \times m$ real matrices. $X > 0$ means that the real symmetric matrix X is positive definite. Symbol $*$ stands for the symmetric block matrices and the superscript T denotes the transpose. I and 0 are identity and zero matrices, respectively, with compatible dimensions.

2. Problem formulation and preliminary knowledge

The following FM-II model is used to describe a 2-D system with fault in the state equation and measurement equation:

$$\begin{cases} x(i+1, j+1) = A_1x(i, j+1) + A_2x(i+1, j) \\ \quad +B_1u(i, j+1) + B_2u(i+1, j) \\ +M_1f(i, j+1) + M_2f(i+1, j) \\ y(i, j) = Cx(i, j) + f_s(i, j) \end{cases} \tag{1}$$

with the following unknown boundary conditions:

$$\begin{aligned} &\sup \|x(i, 0)\| < \infty, i = 0, 1, \ldots; \sup \|x(0, j)\| < \infty, j = 1, 2, \ldots \\ &\sup \|f(i, 0)\| < \infty, i = 0, 1, \ldots; \sup \|f(0, j)\| < \infty, j = 1, 2, \ldots \\ &\sup \|f_s(i, 0)\| < \infty, i = 0, 1, \ldots; \sup \|f_s(0, j)\| < \infty, j = 1, 2, \ldots \end{aligned} \tag{2}$$

where $x(i, j) \in R^n$, $y(i, j) \in R^p$ and $u(i, j) \in R^m$ are the system state vector, measurement output vector and input vector, respectively; $f(i, j) \in R^q$, the fault in the state equation, could represent the actuator fault or process fault; $f_s(i, j) \in R^p$, the fault in the measurement equation, represents the sensor fault; $C, A_k, B_k, M_k, (k = 1, 2)$ are system matrices with appropriate dimensions.

As the fault description that is given in 1-D systems (Dong, Wang, Ding, & Gao, 2014), the dynamic characteristics of $f(i, j) \in R^q$ can be given as follows:

$$f(i+1, j+1) = A_{d1}f(i, j+1) + A_{d2}f(i+1, j) \tag{3}$$

where A_{d1}, A_{d2} are known matrices with appropriate dimensions.

Remark 2.1: The proposed fault modelled by (3) may occur in a probabilistic way based on an individual probability distribution. In practice, by analysing the historical faults data or using the established fault dynamic characteristics of some common faults, prior knowledge of faults may be easy to obtain and it will simplify our analysis. For example, if we know that the 2-D system fault is a constant fault, the fault can be described as follows:

$$f(i+1, j+1) = 0.5f(i, j+1) + 0.5f(i+1, j)$$

The objective is to design the observer to estimate the system state, fault in the state equation and fault in the measurement equation simultaneously. Define $\bar{x}(i+1, j+1) = [\,x^T(i+1, j+1)\;\; f_s^T(i+1, j+1)\,]^T$ and consider the following observer:

$$\begin{cases} z(i+1, j+1) = F_1z(i, j+1) + F_2z(i+1, j) \\ \quad +H_1u(i, j+1) + H_2u(i+1, j) \\ +G_1y(i, j+1) + G_2y(i+1, j) + R_1\hat{f}(i, j+1) + R_2\hat{f}(i+1, j) \\ \hat{f}(i+1, j+1) = A_{d1}\hat{f}(i, j+1) + A_{d2}\hat{f}(i+1, j) \\ -K_{d1}\bar{C}z(i, j+1) - K_{d2}\bar{C}z(i+1, j) \\ \hat{\bar{x}}(i, j) = z(i, j) + Ty(i, j) \end{cases} \tag{4}$$

with the following unknown boundary conditions:

$$\sup \|z(i, 0)\| < \infty, i = 0, 1, \ldots; \sup \|z(0, j)\| < \infty, j = 1, 2, \ldots \tag{5}$$

where $z(i, j) \in R^{n+p}$ and $\hat{\bar{x}}(i, j)$ are, respectively, the state and output of the observer, $\hat{\bar{x}}(i, j)$ and $\hat{f}(i, j)$ are, respectively, the estimation of $\bar{x}(i, j)$ and $f(i, j)$. $F_k, H_k, G_k, L_k, K_{dk}, (k = 1, 2)$ are the matrices that need to be designed. Define $e(i, j) = \bar{x}(i, j) - \hat{\bar{x}}(i, j)$, $e_d(i, j) = f(i, j) - \hat{f}(i, j)$, then introduce the definition of the 2-D observer.

Definition 2.1 (Zhao et al., 2015)**:** The observer (4) is an asymptotic observer for system (1), if $\lim_{i,j\to\infty} e(i, j) = \mathbf{0}$ and

$\lim_{i,j\to\infty} e_d(i,j) = \mathbf{0}$ hold for any system boundary conditions satisfying (2), any observer boundary conditions satisfying (5) and any input sequence $u(i,j)$.

Definition 2.2 (Zhao et al., 2015)**:** The observer (4) is a uniformly ultimately bounded observer for system (1), if there exists $\zeta > 0$ such that for any system boundary conditions satisfying (2), any observer boundary conditions satisfying (5) and any input sequence $u(i,j)$, there exist $I > 0$ and $J > 0$, such that $\|e(i,j)\| \le \zeta$, $\|e_d(i,j)\| \le \zeta$, $\forall i \ge I, \forall j \ge J$.

Lemma 2.1 (*Hinamoto, 1997*)**:** *The 2-D system (1) is asymptotically stable if there exists a positive definite and radially unbounded function* $V(i,j)$: $R^n \to R$, *and* $\alpha, \beta > 0$ *with* $\alpha + \beta = 1$ *such that*

$$\Delta V(i,j) = V(x(i+1,j+1)) - \alpha V(x(i,j+1)) - \beta V(x(i+1,j)) < 0 \tag{6}$$

for all $[\,x^T(i,j+1)\ x^T(i+1,j)\,]^T \ne 0$.

Note that Lemma 2.1 presents a sufficient condition for asymptotic stability for the 2-D system (1), and the condition holds for one pair of (α,β) satisfying $\alpha,\ \beta > 0$ and $\alpha+\beta = 1$.

Lemma 2.2 (*Boyd, El Ghaoui, Feron, & Balakrishnan, 1994*)**:** *For any matrices or vectors* w, h *and any positive definite matrix* X *with appropriate dimensions, the following inequality always holds:*

$$hw + w^T h^T \le hXh^T + w^T X^{-1} w \tag{7}$$

3. Observer design

3.1. Asymptotically stable observer

Define the following matrices:

$$E = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} A_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix},$$
$$\bar{B}_2 = \begin{bmatrix} B_2 \\ \mathbf{0} \end{bmatrix}, \bar{C} = [\,C \quad I_p\,], \bar{M}_1 = \begin{bmatrix} M_1 \\ \mathbf{0} \end{bmatrix}, \bar{M}_2 = \begin{bmatrix} M_2 \\ \mathbf{0} \end{bmatrix} \tag{8}$$

One can obtain the following augmented 2-D singular systems:

$$\begin{cases} E\bar{x}(i+1,j+1) = \bar{A}_1\bar{x}(i,j+1) + \bar{A}_2\bar{x}(i+1,j) \\ \quad +\bar{B}_1 u(i,j+1) + \bar{B}_2 u(i+1,j) \\ +\bar{M}_1 f(i,j+1) + \bar{M}_2 f(i+1,j) \\ y(i,j) = \bar{C}\bar{x}(i,j) \\ f(i+1,j+1) = A_{d1} f(i,j+1) + A_{d2} f(i+1,j) \end{cases} \tag{9}$$

the boundary conditions of the augmented systems are:

$$\begin{aligned} &\sup \|\bar{x}(i,0)\| < \infty, i = 0,1,\ldots; \sup \|\bar{x}(0,j)\| < \infty, j = 1,2,\ldots \\ &\sup \|f(i,0)\| < \infty, i = 0,1,\ldots; \sup \|f(0,j)\| < \infty, j = 1,2,\ldots \end{aligned} \tag{10}$$

Theorem 3.1: *For 2-D system (9) and the following observer matrices:*

$$\begin{aligned} &L, K_1, K_2 \in R^{(n+p)\times p}, \quad K_{d1}, K_{d1} \in R^{q\times p} \\ &F_1 = (E+L\bar{C})^{-1}(\bar{A}_1 - K_1\bar{C}), F_2 = (E+L\bar{C})^{-1}(\bar{A}_2 - K_2\bar{C}) \\ &H_1 = (E+L\bar{C})^{-1}\bar{B}_1, H_2 = (E+L\bar{C})^{-1}\bar{B}_2, T = (E+L\bar{C})^{-1}L \\ &R_1 = (E+L\bar{C})^{-1}\bar{M}_1, R_2 = (E+L\bar{C})^{-1}\bar{M}_2 \\ &G_1 = (E+L\bar{C})^{-1}\bar{A}_1(E+L\bar{C})^{-1}L, G_2 \\ &\quad = (E+L\bar{C})^{-1}\bar{A}_2(E+L\bar{C})^{-1}L \end{aligned} \tag{11}$$

and the boundary conditions

$$\begin{aligned} &\sup \|\hat{\bar{x}}(i,0)\| < \infty,\ i = 0,1,\ldots; \quad \sup \|\hat{\bar{x}}(0,j)\| < \infty,\ j = 1,2,\ldots \\ &\sup \|\hat{f}(i,0)\| < \infty,\ i = 0,1,\ldots; \quad \sup \|\hat{f}(0,j)\| < \infty,\ j = 1,2,\ldots \end{aligned} \tag{12}$$

If there exist a positive definite matrix $P \in R^{(n+p+q)\times(n+p+q)}$, matrices $Y_1,\ Y_2 \in R^{(n+p+q)\times p}$ and positive scalars $\alpha, \beta,\ (\alpha+\beta = 1)$ such that the following linear matrix inequality:

$$\begin{bmatrix} -\alpha P & * & * \\ \mathbf{0} & -\beta P & * \\ P(\tilde{E}_L)^{-1}\tilde{A}_1 - Y_1\tilde{C} & P(\tilde{E}_L)^{-1}\tilde{A}_2 - Y_2\tilde{C} & -P \end{bmatrix} < 0 \tag{13}$$

$$\tilde{K}_1 = (P\tilde{E}_L^{-1})^{-1}Y_1, \tilde{K}_2 = (P\tilde{E}_L^{-1})^{-1}Y_2 \tag{14}$$

holds, then $\lim_{i,j\to\infty}(\bar{x}(i,j) - \hat{\bar{x}}(i,j)) = \mathbf{0}$ and $\lim_{i,j\to\infty}(f(i,j) - \hat{f}(i,j)) = \mathbf{0}$. The definitions of $\tilde{K}_1, \tilde{K}_2, \tilde{E}_L$ are given in (20).

Proof: Partitioning $L = [\,L_1^T\ L_2^T\,]^T$, where $L_1 \in R^{n\times p}, L_2 \in R^{p\times p}$, owing to

$$\begin{aligned} \mathrm{rank}\,(E+L\bar{C}) &= \mathrm{rank}\left(\begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}[\,C \quad I_p\,]\right) \\ &= \mathrm{rank}\begin{bmatrix} I_n + L_1C & L_1 \\ L_2C & L_2 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} I_n & L_1 \\ \mathbf{0} & L_2 \end{bmatrix} = \mathrm{rank}\,(L_2) + n \end{aligned} \tag{15}$$

■

It is obvious that if and only if the square matrix L_2 is of full rank, $(E+L\bar{C})$ is nonsingular. Hence, letting L_2 be nonsingular, the following relationship can be obtained:

$$\begin{aligned} \bar{C}(E+L\bar{C})^{-1}L &= [\,C \quad I_p\,]\begin{bmatrix} I_n & -L_1(L_2)^{-1} \\ -C & (I_p + CL_1)(L_2)^{-1} \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \\ &= [\,C \quad I_p\,]\begin{bmatrix} \mathbf{0} \\ I_p \end{bmatrix} = I_p \end{aligned} \tag{16}$$

Adding $Ly(i+1, j+1)$ to both sides of the first equation in (9) yields

$$\begin{aligned}(E+L\bar{C})\bar{x}(i+1,j+1) &= \bar{A}_1\bar{x}(i,j+1)+\bar{A}_2\bar{x}(i+1,j)\\ &+\bar{B}_1u(i,j+1)+\bar{B}_2u(i+1,j)\\ &+\bar{M}_1f(i,j+1)+\bar{M}_2f(i+1,j)+Ly(i+1,j+1)\\ &=(\bar{A}_1-K_1\bar{C})\bar{x}(i,j+1)+(\bar{A}_2-K_2\bar{C})\bar{x}(i+1,j)\\ &+K_1y(i,j+1)+K_2y(i+1,j)\\ &+\bar{B}_1u(i,j+1)+\bar{B}_2u(i+1,j)+Ly(i+1,j+1)\\ &+\bar{M}_1f(i,j+1)+\bar{M}_2f(i+1,j)\end{aligned} \tag{17}$$

Substituting $z(i, j) = \hat{\bar{x}}(i, j) - (E + L\bar{C})^{-1}Ly(i, j)$ into the observer shown in (4), using Equations (11) and (16), the observer in (4) becomes:

$$\begin{cases}(E+L\bar{C})\hat{\bar{x}}(i+1,j+1)=(\bar{A}_1-K_1\bar{C})\hat{\bar{x}}(i,j+1)\\ \quad+(\bar{A}_2-K_2\bar{C})\hat{\bar{x}}(i+1,j)\\ +\bar{B}_1u(i,j+1)+\bar{B}_2u(i+1,j)+K_1y(i,j+1)\\ +K_2y(i+1,j)+Ly(i+1,j+1)\\ \quad+\bar{M}_1\hat{f}(i,j+1)+\bar{M}_2\hat{f}(i+1,j)\\ \hat{f}(i+1,j+1)=A_{d1}\hat{f}(i,j+1)+A_{d2}\hat{f}(i+1,j)\\ \quad-K_{d1}\bar{C}z(i,j+1)-K_{d2}\bar{C}z(i+1,j)\\ =A_{d1}\hat{f}(i,j+1)+A_{d2}\hat{f}(i+1,j)\\ \quad-K_{d1}\bar{C}\left(\hat{\bar{x}}(i,j+1)-(E+L\bar{C})^{-1}Ly(i,j+1)\right)\\ -K_{d2}\bar{C}\left(\hat{\bar{x}}(i+1,j)-(E+L\bar{C})^{-1}Ly(i+1,j)\right)\\ =A_{d1}\hat{f}(i,j+1)+A_{d2}\hat{f}(i+1,j)\\ \quad-K_{d1}\bar{C}\left(\hat{\bar{x}}(i,j+1)-(E+L\bar{C})^{-1}L\bar{C}\bar{x}(i,j+1)\right)\\ -K_{d2}\bar{C}\left(\hat{\bar{x}}(i+1,j)-(E+L\bar{C})^{-1}L\bar{C}\bar{x}(i+1,j)\right)\\ =\hat{f}(i+1,j+1)=A_{d1}\hat{f}(i,j+1)+A_{d2}\hat{f}(i+1,j)\\ \quad+K_{d1}\bar{C}[\bar{x}(i,j+1)-\hat{\bar{x}}(i,j+1)]\\ +K_{d2}\bar{C}[\bar{x}(i+1,j)-\hat{\bar{x}}(i+1,j)]\end{cases} \tag{18}$$

Define $e(i, j) = \bar{x}(i, j) - \hat{\bar{x}}(i, j)$ and $e_d(i, j) = f(i, j) - \hat{f}(i, j)$. It comes the following error dynamic system:

$$\begin{cases}(E+L\bar{C})e(i+1,j+1)=(\bar{A}_1-K_1\bar{C})e(i,j+1)\\ \quad+(\bar{A}_2-K_2\bar{C})e(i+1,j)\\ +\bar{M}_1e_d(i,j+1)+\bar{M}_2e_d(i+1,j)\\ e_d(i+1,j+1)=A_{d1}e_d(i,j+1)\\ \quad+A_{d2}e_d(i+1,j)-K_{d1}\bar{C}e(i,j+1)-K_{d2}\bar{C}e(i+1,j)\end{cases} \tag{19}$$

Rewrite (19) into a compact form:

$$\underbrace{\begin{bmatrix}E+L\bar{C} & \mathbf{0}\\ \mathbf{0} & I_q\end{bmatrix}}_{\tilde{E}_L}\underbrace{\begin{bmatrix}e(i+1,j+1)\\ e_d(i+1,j+1)\end{bmatrix}}_{\tilde{e}(i+1,j+1)}=\left(\underbrace{\begin{bmatrix}\bar{A}_1 & \bar{M}_1\\ \mathbf{0} & A_{d1}\end{bmatrix}}_{\tilde{A}_1}-\underbrace{\begin{bmatrix}K_1\\ K_{d1}\end{bmatrix}}_{\tilde{K}_1}\underbrace{\begin{bmatrix}\bar{C} & \mathbf{0}\end{bmatrix}}_{\tilde{C}}\right)\underbrace{\begin{bmatrix}e(i,j+1)\\ e_d(i,j+1)\end{bmatrix}}_{\tilde{e}(i,j+1)}+\left(\underbrace{\begin{bmatrix}\bar{A}_2 & \bar{M}_2\\ \mathbf{0} & A_{d2}\end{bmatrix}}_{\tilde{A}_2}-\underbrace{\begin{bmatrix}K_2\\ K_{d2}\end{bmatrix}}_{\tilde{K}_2}\underbrace{\begin{bmatrix}\bar{C} & \mathbf{0}\end{bmatrix}}_{\tilde{C}}\right)\underbrace{\begin{bmatrix}e(i+1,j)\\ e_d(i+1,j)\end{bmatrix}}_{\tilde{e}(i+1,j)} \tag{20}$$

Furthermore, it can be rewritten as

$$\begin{aligned}&\tilde{e}(i+1,j+1)\\ &=(\tilde{E}_L)^{-1}(\tilde{A}_1-\tilde{K}_1\tilde{C})\tilde{e}(i,j+1)+(\tilde{E}_L)^{-1}(\tilde{A}_2-\tilde{K}_2\tilde{C})\tilde{e}(i+1,j)\\ &=\begin{bmatrix}\tilde{F}_1 & \tilde{F}_2\end{bmatrix}\begin{bmatrix}\tilde{e}(i,j+1)\\ \tilde{e}(i+1,j)\end{bmatrix}=\tilde{F}\tilde{e}\end{aligned} \tag{21}$$

According to Lemma 2.1, let P be a positive definite symmetric matrix and choose $V = \tilde{e}(i+1, j+1)^T P\tilde{e}(i+1, j+1)$ as the radially unbounded Lyapunov function. Then, one can get:

$$\begin{aligned}\Delta V &= \tilde{e}(i+1,j+1)^TP\tilde{e}(i+1,j+1)\\ &-\alpha\tilde{e}(i,j+1)^TP\tilde{e}(i,j+1)-\beta\tilde{e}(i+1,j)^TP\tilde{e}(i+1,j)\\ &\alpha,\beta>0,\alpha+\beta=1\end{aligned} \tag{22}$$

Taking (21) into consideration, it yields:

$$\begin{aligned}\Delta V &= [\tilde{F}\tilde{e}]^TP[\tilde{F}\tilde{e}]-\alpha\tilde{e}^T(i,j+1)P\tilde{e}(i,j+1)\\ &-\beta\tilde{e}^T(i+1,j)P\tilde{e}(i+1,j)\\ &=\tilde{e}^T(\tilde{F}^TP\tilde{F}-\begin{bmatrix}\alpha P & \mathbf{0}\\ \mathbf{0} & \beta P\end{bmatrix})\tilde{e}\end{aligned} \tag{23}$$

According to Equation (13) and Schur complement, one can get:

$$\tilde{F}^TP\tilde{F}-\begin{bmatrix}\alpha P & \mathbf{0}\\ \mathbf{0} & \beta P\end{bmatrix}<0 \tag{24}$$

Referring to Lemma 2.1, the error dynamic system (21) is asymptotically stable or $\lim_{i,j\to\infty}\tilde{e}(i+1, j+1) = 0$. Referring to Definition 2.1, it completes the proof.

Remark 3.1: The observer (4) is a modified form of the descriptor observer, but $E + L\bar{C}$ is nonsingular. It is obvious that observer (4) is equal to observer (18), but (18) requires knowledge of future output $y(i+1, j+1)$, which may cause troubles in online implementation. Therefore, observer (4) is more tractable.

Remark 3.2: Notice that the description in (4) is similar to the description about the unknown input system in Bisiacco and Elena Valcher (2004) and Xu, Lin, Makur, and Xu (2011), but an additional asymptotic estimation of fault in the state equation and measurement equation can be obtained from observer (4) by comparing with the observers presented in Bisiacco and Elena Valcher (2004) and Xu et al. (2011). The fault vectors f and f_s could be any form and even be unbounded.

3.2. Uniformly ultimately bounded observer

Considering the effect of disturbance on the fault model, the dynamic characteristics of $f(i, j) \in R^q$ can be given as follows:

$$\begin{aligned}f(i+1,j+1)&=A_{d1}f(i,j+1)+A_{d2}f(i+1,j)\\ &+h(i,j+1)+h(i+1,j)\end{aligned} \tag{25}$$

where A_{d1}, A_{d2} are known matrices with appropriate dimensions and $h(i, j)$ is the disturbance. The fault in the measurement equation $f_s(i, j)$ and the disturbance $h(i, j)$ satisfy: $\|f_s(i, j)\| \le \nu$, $\|h(i, j)\| \le \theta$.

Define the following matrices:

$$E=\begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{A}_{1\text{new}}=\begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & -I_P \end{bmatrix}, \bar{A}_{2\text{new}}=\begin{bmatrix} A_2 & \mathbf{0} \\ \mathbf{0} & -I_P \end{bmatrix},$$
$$\bar{B}_1=\begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix}, \bar{B}_2=\begin{bmatrix} B_2 \\ \mathbf{0} \end{bmatrix},$$
$$N_1=N_2=\begin{bmatrix} \mathbf{0} \\ I_P \end{bmatrix}, \bar{C}=\begin{bmatrix} C & I_p \end{bmatrix}, \bar{M}_1=\begin{bmatrix} M_1 \\ \mathbf{0} \end{bmatrix}, \bar{M}_2=\begin{bmatrix} M_2 \\ \mathbf{0} \end{bmatrix} \quad (26)$$

One can obtain the following augment 2-D singular systems:

$$\begin{cases} E\bar{x}(i+1,\,j+1)=\bar{A}_{1\text{new}}\bar{x}(i,\,j+1)+\bar{A}_{2\text{new}}\bar{x}(i+1,\,j) \\ +\bar{B}_1u(i,\,j+1)+\bar{B}_2u(i+1,\,j) \\ +\bar{M}_1f(i,\,j+1)+\bar{M}_2f(i+1,\,j) \\ +N_1f_s(i,\,j+1)+N_2f_s(i+1,\,j) \\ y(i,\,j)=\bar{C}\bar{x}(i,\,j) \\ f(i+1,\,j+1)=A_{d1}f(i,\,j+1) \\ +A_{d2}f(i+1,\,j)+h(i,\,j+1)+h(i+1,\,j) \end{cases} \quad (27)$$

The boundary conditions of the augmented systems are:

$$\begin{aligned} &\sup\|\bar{x}(i,0)\|<\infty,\ i=0,1,\ldots;\ \sup\|\bar{x}(0,j)\|<\infty,\ j=1,2,\ldots \\ &\sup\|f(i,0)\|<\infty,\ i=0,1,\ldots;\ \sup\|f(0,j)\|<\infty,\ j=1,2,\ldots \\ &\sup\|f_s(i,0)\|<\infty,\ i=0,1,\ldots;\ \sup\|f_s(0,j)\|<\infty,\ j=1,2,\ldots \end{aligned} \quad (28)$$

Theorem 3.2: *For system (27) with boundary condition (28), let L be a matrix such that $E+L\bar{C}$ is nonsingular. If there exist a positive definite matrix $P\in R^{(n+p+q)\times(n+p+q)}$, matrices $Y_1, Y_2\in R^{(n+p+q)\times p}$ and positive scalars α, β, $\delta(\alpha+\beta=1)$, the following linear matrix inequality (LMI):*

$$\begin{bmatrix} \delta I-\alpha P & * & * \\ \mathbf{0} & \delta I-\beta P & * \\ P(\tilde{E}_L)^{-1}\tilde{A}_{1\text{new}}-Y_1\tilde{C} & P(\tilde{E}_L)^{-1}\tilde{A}_{2\text{new}}-Y_2\tilde{C} & \delta I-P \end{bmatrix}<0 \quad (29)$$

$$\tilde{K}_1=(P\tilde{E}_L^{-1})^{-1}Y_1,\ \tilde{K}_2=(P\tilde{E}_L^{-1})^{-1}Y_2 \quad (30)$$

holds, then (4) is a uniformly ultimately bounded observer for system (27). The observer gain matrices are given in (11) of Theorem 3.1. The uniformly ultimately bounded error is given as follows:

$$\|\tilde{e}(i,\,j)\|_2<\sqrt{\frac{\lambda_{\max}(\Xi)\|\upsilon\|_2{}^2+\lambda_{\max}(\Pi)\|\theta\|_2{}^2}{\lambda_{\min}(\Omega)}},\ \forall i>I,\ j>J \quad (31)$$

The relevant matrices are defined in (35) and (44).

Proof: For Equation (27), add $Ly(i+1,\,j+1)$ to both sides, one can get:

$$\begin{cases} (E+L\bar{C})x(i+1,\,j+1)=(\bar{A}_{1\text{new}}-K_1\bar{C})x(i,\,j+1) \\ \quad +(\bar{A}_{2\text{new}}-K_2\bar{C})x(i+1,\,j) \\ +\bar{B}_1u(i,\,j+1)+\bar{B}_2u(i+1,\,j)+\bar{M}_1f(i,\,j+1) \\ \quad +\bar{M}_2f(i+1,\,j)+N_1f_s(i,\,j+1)+N_2f_s(i+1,\,j) \\ +Ly(i+1,\,j+1)+K_1y(i,\,j+1)+K_2y(i+1,\,j) \\ f(i+1,\,j+1)=A_{d1}f(i,\,j+1)+A_{d2}f(i+1,\,j) \\ \quad +h(i,\,j+1)+h(i+1,\,j) \end{cases} \quad (32)$$

■

Substitute $z(i,\,j)=\hat{\bar{x}}(i,\,j)-(E+L\bar{C})^{-1}Ly(i,\,j)$ into (4), using Equations (11) and (16), the observer in (4) becomes:

$$\begin{cases} (E+L\bar{C})\hat{\bar{x}}(i+1,\,j+1) \\ =(\bar{A}_{1\text{new}}-K_1\bar{C})\hat{\bar{x}}(i,\,j+1)+(\bar{A}_{2\text{new}}-K_2\bar{C})\hat{\bar{x}}(i+1,\,j) \\ +\bar{B}_1u(i,\,j+1)+\bar{B}_2u(i+1,\,j)+K_1y(i,\,j+1) \\ +K_2y(i+1,\,j)+Ly(i+1,\,j+1) \\ +\bar{M}_1\hat{f}(i,\,j+1)+\bar{M}_2\hat{f}(i+1,\,j) \\ \hat{f}(i+1,\,j+1)=A_{d1}\hat{f}(i,\,j+1)+A_{d2}\hat{f}(i+1,\,j) \\ +K_{d1}\bar{C}[\bar{x}(i,\,j+1)-\hat{\bar{x}}(i,\,j+1)] \\ +K_{d2}\bar{C}[\bar{x}(i+1,\,j)-\hat{\bar{x}}(i+1,\,j)] \end{cases} \quad (33)$$

Define $e(i,\,j)=\bar{x}(i,\,j)-\hat{\bar{x}}(i,\,j)$ and $e_d(i,\,j)=f(i,\,j)-\hat{f}(i,\,j)$, thus, the error dynamic system can be obtained as follows:

$$\begin{cases} (E+L\bar{C})e(i+1,\,j+1)=(\bar{A}_{1\text{new}}-K_1\bar{C})e(i,\,j+1) \\ +(\bar{A}_{2\text{new}}-K_2\bar{C})e(i+1,\,j) \\ +N_1f_s(i,\,j+1)+N_2f_s(i+1,\,j) \\ +\bar{M}_1e_d(i,\,j+1)+\bar{M}_2e_d(i+1,\,j) \\ e_d(i+1,\,j+1)=A_{d1}e_d(i,\,j+1)+A_{d2}e_d(i+1,\,j) \\ -K_d\bar{C}[e(i,\,j+1)+e(i+1,\,j)] \\ +h(i,\,j+1)+h(i+1,\,j) \end{cases} \quad (34)$$

The error dynamic system (34) can be presented in a compact form as

$$\begin{aligned} &\underbrace{\begin{bmatrix} E+L\bar{C} & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}}_{\tilde{E}_L}\underbrace{\begin{bmatrix} e(i+1,\,j+1) \\ e_d(i+1,\,j+1) \end{bmatrix}}_{\tilde{e}(i+1,j+1)} \\ &=\left(\underbrace{\begin{bmatrix} \bar{A}_{1\text{new}} & \bar{M}_1 \\ \mathbf{0} & A_{d1} \end{bmatrix}}_{\tilde{A}_{1\text{new}}}-\underbrace{\begin{bmatrix} K_1 \\ K_{d1} \end{bmatrix}}_{\tilde{K}_1}\underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}}\right)\begin{bmatrix} e(i,\,j+1) \\ e_d(i,\,j+1) \end{bmatrix} \\ &+\left(\underbrace{\begin{bmatrix} \bar{A}_{2\text{new}} & \bar{M}_2 \\ \mathbf{0} & A_{d2} \end{bmatrix}}_{\tilde{A}_{2\text{new}}}-\underbrace{\begin{bmatrix} K_2 \\ K_{d2} \end{bmatrix}}_{\tilde{K}_2}\underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}}\right)\begin{bmatrix} e(i+1,\,j) \\ e_d(i+1,\,j) \end{bmatrix} \\ &+\underbrace{\begin{bmatrix} N_1 & N_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\tilde{N}}\begin{bmatrix} f_s(i,\,j+1) \\ f_s(i+1,\,j) \end{bmatrix}+\underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_q & I_q \end{bmatrix}}_{\tilde{Q}}\begin{bmatrix} h(i,\,j+1) \\ h(i+1,\,j) \end{bmatrix} \end{aligned} \quad (35)$$

Furthermore, (35) can be rewritten as

$$\begin{aligned}&\tilde{e}(i+1,j+1)=(\tilde{E}_L)^{-1}(\tilde{A}_{1\text{new}}-\tilde{K}_1\tilde{C})\tilde{e}(i,j+1)\\&\quad+(\tilde{E}_L)^{-1}(\tilde{A}_{2\text{new}}-\tilde{K}_2\tilde{C})\tilde{e}(i+1,j)\\&\quad+(\tilde{E}_L)^{-1}\tilde{N}\begin{bmatrix}f_s(i,j+1)\\f_s(i+1,j)\end{bmatrix}+(\tilde{E}_L)^{-1}\tilde{Q}\begin{bmatrix}h(i,j+1)\\h(i+1,j)\end{bmatrix}\\&=[\widehat{F}_1\ \widehat{F}_2]\begin{bmatrix}\tilde{e}(i,j+1)\\\tilde{e}(i+1,j)\end{bmatrix}+\widehat{N}\begin{bmatrix}f_s(i,j+1)\\f_s(i+1,j)\end{bmatrix}+\widehat{Q}\begin{bmatrix}h(i,j+1)\\h(i+1,j)\end{bmatrix}\\&=\widehat{F}\tilde{e}+\widehat{N}f_s+\widehat{Q}h\end{aligned} \tag{36}$$

To get uniformly ultimately bounded estimation of system (27), the open-loop of the error system (36) should be asymptotically stable. However, owing to the existence of faults in the measurement equation and state equation, the error system (36) cannot be asymptotically stable. Next, the effects of faults and disturbance on the estimation error are further analysed.

According to Lemma 2.1, let P be a positive definite symmetric matrix and choose $V=\tilde{e}(i+1,j+1)^TP\tilde{e}(i+1,j+1)$ as the radially unbounded Lyapunov function. Then, one can get:

$$\begin{aligned}&\Delta V=\tilde{e}(i+1,j+1)^TP\tilde{e}(i+1,j+1)\\&\quad-\alpha\tilde{e}(i,j+1)^TP\tilde{e}(i,j+1)-\beta\tilde{e}(i+1,j)^TP\tilde{e}(i+1,j)<0\\&\quad\alpha,\beta>0,\alpha+\beta=1\end{aligned} \tag{37}$$

Taking (36) into consideration, it yields:

$$\begin{aligned}\Delta V&=[\widehat{F}\tilde{e}+\widehat{N}f_s+\widehat{Q}h]^TP[\widehat{F}\tilde{e}+\widehat{N}v+\widehat{Q}h]\\&\quad-\alpha\tilde{e}^T(i,j+1)P\tilde{e}(i,j+1)-\beta\tilde{e}^T(i+1,j)P\tilde{e}(i+1,j)\\&=\tilde{e}^T(\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix})\tilde{e}+f_s^T\widehat{N}^TP\widehat{N}f_s\\&\quad+h^T\widehat{Q}^TP\widehat{Q}h+2\tilde{e}^T\widehat{F}^TP\widehat{N}f_s+2\tilde{e}^T\widehat{F}^TP\widehat{Q}h+2f_s^T\widehat{N}^TP\widehat{Q}h\end{aligned} \tag{38}$$

Using the Schur complement lemma for (29), it can be found that

$$\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix}<0 \tag{39}$$

There must exist a positive scalar δ that satisfies:

$$\delta I+\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix}<0 \tag{40}$$

One of δ can be chosen as

$$\delta=-0.1\sigma_{\max}\left(\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix}\right) \tag{41}$$

Referring to Lemma 2.2, one can get:

$$\begin{aligned}&2\tilde{e}^T\widehat{F}^TP\widehat{N}f_s\le\tilde{e}^T\delta_1I\tilde{e}+f_s^T\widehat{N}^TP\widehat{F}\delta_1^{-1}I\widehat{F}^TP\widehat{N}f_s\\&2\tilde{e}^T\widehat{F}^TP\widehat{Q}h\le\tilde{e}^T\delta_2I\tilde{e}+h^T\widehat{Q}^TP\widehat{F}\delta_2^{-1}I\widehat{F}^TP\widehat{Q}h\\&2f_s^T\widehat{N}^TP\widehat{Q}h\le f_s^T\delta_3If_s+h^T\widehat{Q}^TP\widehat{N}\delta_3^{-1}I\widehat{N}^TP\widehat{Q}h\end{aligned} \tag{42}$$

Then, (38) can be rewritten as

$$\begin{aligned}\Delta V&=\tilde{e}^T\left(\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix}\right)\tilde{e}+f_s^T\widehat{N}^TP\widehat{N}f_s+h^T\widehat{Q}^TP\widehat{Q}h\\&\quad+2\tilde{e}^T\widehat{F}^TP\widehat{N}f_s+2\tilde{e}^T\widehat{F}^TP\widehat{Q}h+2f_s^T\widehat{N}^TP\widehat{Q}h\\&\le\tilde{e}^T\left(\delta I+\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix}\right)\tilde{e}\\&\quad+f_s^T(\widehat{N}^TP\widehat{N}+\widehat{N}^TP\widehat{F}\delta_1^{-1}I\widehat{F}^TP\widehat{N}+\delta_3I)f_s\\&\quad+h^T(\widehat{Q}^TP\widehat{Q}+\widehat{Q}^TP\widehat{F}\delta_2^{-1}I\widehat{F}^TP\widehat{Q}+\widehat{Q}^TP\widehat{N}\delta_3^{-1}I\widehat{N}^TP\widehat{Q})h\\\delta&=\delta_1+\delta_2\end{aligned} \tag{43}$$

Obviously, when $\tilde{e}$ is more than a certain definite value, $\Delta V<0$; otherwise, when $\tilde{e}$ is less than a certain definite value, $\Delta V\ge 0$.

$$\begin{aligned}&\Delta V\\&\le\tilde{e}^T\left(\delta I+\widehat{F}^TP\widehat{F}-\begin{bmatrix}\alpha P&\mathbf{0}\\\mathbf{0}&\beta P\end{bmatrix}\right)\tilde{e}\\&+f_s^T\left(\widehat{N}^TP\widehat{N}+\widehat{N}^TP\widehat{F}\delta_1^{-1}I\widehat{F}^TP\widehat{N}+\delta_3I\right)f_s\\&+h^T\left(\widehat{Q}^TP\widehat{Q}+\widehat{Q}^TP\widehat{F}\delta_2^{-1}I\widehat{F}^TP\widehat{Q}+\widehat{Q}^TP\widehat{N}\delta_3^{-1}I\widehat{N}^TP\widehat{Q}\right)h\\&=-\tilde{e}^T\Omega\tilde{e}+f_s^T\Xi f_s+h^T\Pi h\end{aligned} \tag{44}$$

When $\Delta V<0$, according to (44), it comes:

$$\tilde{e}^T\Omega\tilde{e}\ge f_s^T\Xi f_s+h^T\Pi h \tag{45}$$

Let $\lambda_{\min}(*)/\lambda_{\max}(*)$ indicate the minimum/maximum eigenvalue of $*$, and it yields:

$$\begin{aligned}&\lambda_{\min}(\Omega)\left\|\tilde{e}\right\|_2^2>\lambda_{\max}(\Xi)\|v\|_2^2+\lambda_{\max}(\Pi)\|\theta\|_2^2\\&\Rightarrow\left\|\tilde{e}\right\|_2>\sqrt{\frac{\lambda_{\max}(\Xi)\|v\|_2^2+\lambda_{\max}(\Pi)\|\theta\|_2^2}{\lambda_{\min}(\Omega)}}\end{aligned} \tag{46}$$

Referring to Lemma 2.1 and (45), when Equation (46) holds, one knows that the estimation error $\tilde{e}$ is convergent; so, there must exist I,J, for $\forall i>I,\ j>J$,

$$\left\|\tilde{e}(i,j)\right\|_2\le\sqrt{\frac{\lambda_{\max}(\Xi)\|v\|_2^2+\lambda_{\max}(\Pi)\|\theta\|_2^2}{\lambda_{\min}(\Omega)}},\forall i>I,\ j>J \tag{47}$$

This completes the proof.

Remark 3.3: Notice that the matrices $\widehat{N}$ and $\widehat{Q}$ can be given as follows:

$$\widehat{N}=(\tilde{E}_L)^{-1}\tilde{N}=\begin{bmatrix}-L_1(L_2)^{-1}&-L_1(L_2)^{-1}\\\left(I_p+CL_1\right)(L_2)^{-1}&\left(I_p+CL_1\right)(L_2)^{-1}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\widehat{Q}=(\tilde{E}_L)^{-1}\tilde{Q}=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\\I_q&I_q\end{bmatrix}$$

If we choose $L_2^{-1} \to \mathbf{0}$, then $\widehat{N} \to \mathbf{0}$; so, the effect of fault in the measurement equation f_s on $\tilde{e}$ can be designed to be arbitrary small.

Remark 3.4: In Equation (26), $\bar{A}_{1\text{new}}$ and $\bar{A}_{2\text{new}}$ can also be written as

$$\bar{A}_1 = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} A_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Then, the final uniformly ultimately bounded error $\tilde{e}$ will become:

$$\underbrace{\begin{bmatrix} E + L\bar{C} & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}}_{\tilde{E}_L} \underbrace{\begin{bmatrix} e(i+1, j+1) \\ e_d(i+1, j+1) \end{bmatrix}}_{\tilde{e}(i+1,j+1)} = \left(\underbrace{\begin{bmatrix} \bar{A}_1 & \bar{M}_1 \\ \mathbf{0} & A_{d1} \end{bmatrix}}_{\tilde{A}_1} - \underbrace{\begin{bmatrix} K_1 \\ K_{d1} \end{bmatrix}}_{\tilde{K}_1} \underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}} \right) \begin{bmatrix} e(i, j+1) \\ e_d(i, j+1) \end{bmatrix} + \left(\underbrace{\begin{bmatrix} \bar{A}_2 & \bar{M}_2 \\ \mathbf{0} & A_{d2} \end{bmatrix}}_{\tilde{A}_2} - \underbrace{\begin{bmatrix} K_2 \\ K_{d2} \end{bmatrix}}_{\tilde{K}_2} \underbrace{\begin{bmatrix} \bar{C} & \mathbf{0} \end{bmatrix}}_{\tilde{C}} \right) \begin{bmatrix} e(i+1, j) \\ e_d(i+1, j) \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_q & I_q \end{bmatrix}}_{\tilde{Q}} \begin{bmatrix} h(i, j+1) \\ h(i+1, j) \end{bmatrix}$$

It just relates to the disturbance $h(i, j)$. There will be no bounded limit on fault in the measurement equation $f_s(i, j)$.

4. Simulation examples

In this section, some examples are provided to demonstrate the feasibility and effectiveness of the proposed observer design method in the previous theorems.

4.1. Example 1

Consider the 2-D FM-II system, where the system coefficient matrices and fault vectors are given as follows:

$$\begin{aligned} &A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & 0.3 \\ -1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &M_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \\ &f(i, j) = 0.4 f(i-1, j) + 0.2 f(i, j-1) \\ &f_s(i, j) = 10 \sin(i+j), h(i, j) = 0.1 \sin(i+j) \end{aligned} \tag{48}$$

The boundary conditions of the observer in this example are assumed to be zero and the boundary conditions of the error dynamic system are equal to the boundary conditions of the original 2-D system. In the rectangular region, the boundary conditions of error are:

$$\begin{aligned} &e_1(0, j) = \sin(2\pi j/l), e_1(i, 0) = \sin(\pi j/l) \\ &e_2(0, j) = \sin(2\pi j/l), e_2(i, 0) = \sin(2\pi j/l) \\ &e_3(0, j) = \sin(2\pi j/l), e_3(i, 0) = \sin(\pi j/l) \\ &e_4(0, j) = \sin(2\pi j/l), e_4(i, 0) = \sin(2\pi j/l), 0 \le i, j \le l = 40 \end{aligned} \tag{49}$$

Solving LMI (13) with $\alpha = \beta = 0.5$, one gets the following matrices:

$$\begin{aligned} &P = \begin{bmatrix} 9.00 & 0.17 & 0.10 & -16.62 \\ 0.17 & 95.88 & 94.16 & -1.44 \\ 0.10 & 94.16 & 94.32 & 0.85 \\ -16.62 & -1.44 & 0.85 & 508.65 \end{bmatrix}, \\ &\tilde{K}_1 = \begin{bmatrix} -0.0028 \\ -0.0074 \\ -0.0003 \\ 0.0027 \end{bmatrix}, \tilde{K}_2 = \begin{bmatrix} -0.0018 \\ -0.0289 \\ 0.0003 \\ -0.0025 \end{bmatrix} \end{aligned} \tag{50}$$

Solving LMI (29) with $\alpha = \beta = 0.5, \delta = 0.01$, one gets the following matrices:

$$\begin{aligned} &P = \begin{bmatrix} 1.48 & 0.29 & 0.24 & -2.13 \\ 0.29 & 4.71 & 4.33 & -0.88 \\ 0.24 & 4.33 & 4.27 & -0.21 \\ -2.13 & -0.88 & -0.21 & 15.37 \end{bmatrix}, \\ &\tilde{K}_1 = \begin{bmatrix} -0.6498 \\ -15.0876 \\ -1.1987 \\ -0.7252 \end{bmatrix}, \tilde{K}_2 = \begin{bmatrix} -0.6861 \\ -15.0586 \\ -1.1951 \\ -0.7238 \end{bmatrix} \end{aligned} \tag{51}$$

The obtained estimation error $e(i, j)$ $(0 \le i, j \le 40)$ is shown in Figures 1 and 2.$e_1(i, j)$ and $e_2(i.j)$ are about the system states, $e_3(i, j)$ and $e_4(i, j)$ are about the fault in the state equation $f(i, j)$ and fault in the measurement equation $f_s(i, j)$, respectively. From Figure 1, it is easy to see that the estimation error of both system states and two kinds of faults converges to zero quickly. From Figure 2, we can see that the estimation error of both system states and faults converges quickly.

In order to more clearly see that the estimation error is asymptotically stable or uniformly ultimately bounded, for fault estimation error in the state equation in Theorems 3.1 and 3.2, we zoom in on Figures 1(c) and 2(c) to get the magnified images in Figures 3 and 4. It can be found that the fault estimation error in the state equation in Figure 1(c) is asymptotically stable, the fault estimation error in the state equation in Figure 2(c) is uniformly ultimately bounded.

4.2. Example 2

Consider the thermal process described by the following partial differential equation (Kaczorek, 1985):

$$\begin{aligned} &T_x(x, t) = -T_t(x, t) - T(x, t) + bu(x, t) \\ &T(0, t) = l_1(t), T(x, 0) = l_2(x), x \in [0, x_L], t \in [0, \infty] \end{aligned} \tag{52}$$

where $T(x, t)$ is the temperature at space point x and time point t, $u(x, t)$ is the given force function, $T(0, t)$ and $T(x, 0)$ are the boundary and initial conditions, respectively and $l_1(t)$ and $l_2(x)$ are known functions. Equation (52) is usually used

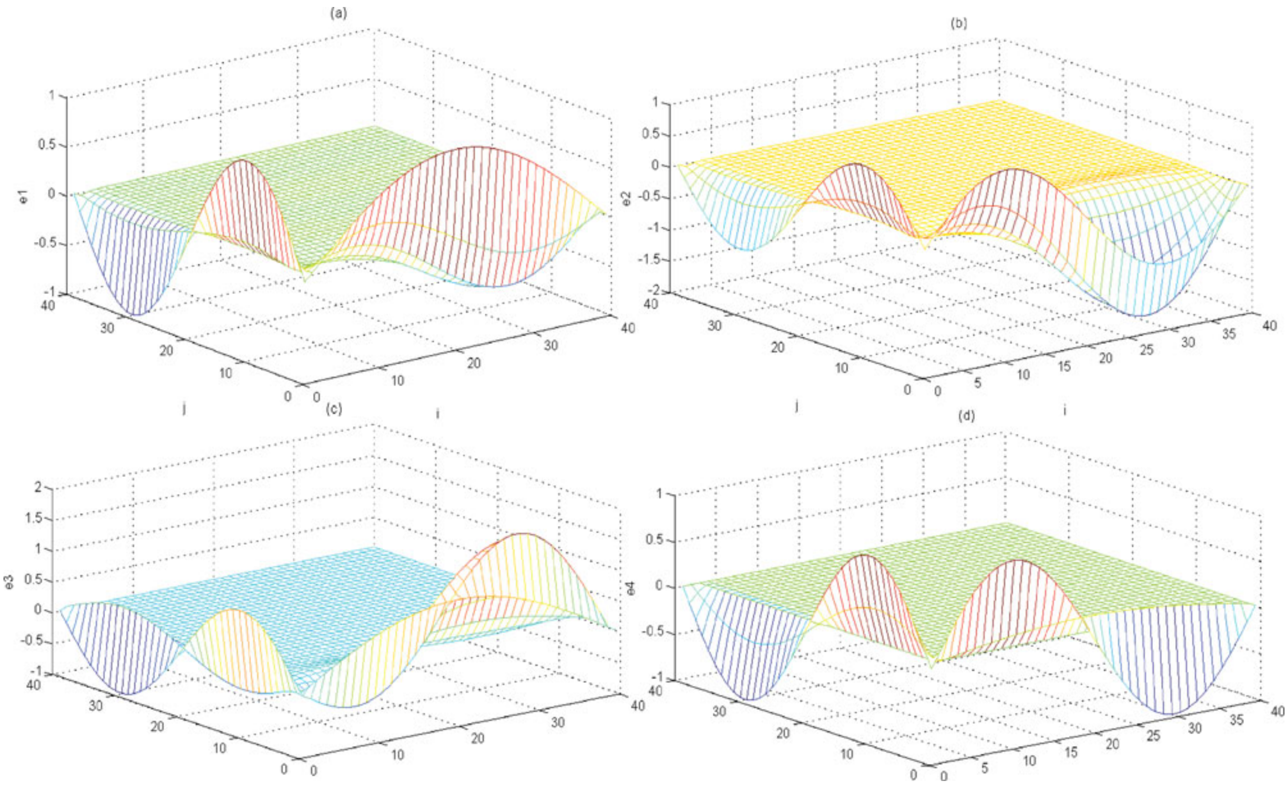

Figure 1. Estimation error $e(i,\ j)$ of the observer in Theorem 3.1 for the 2-D system presented in (48). $e_1(i,\ j)$, $e_2(i.j)$, $e_3(i,\ j)$ and $e_4(i,\ j)$ are shown in subplots (a), (b), (c) and (d), respectively.

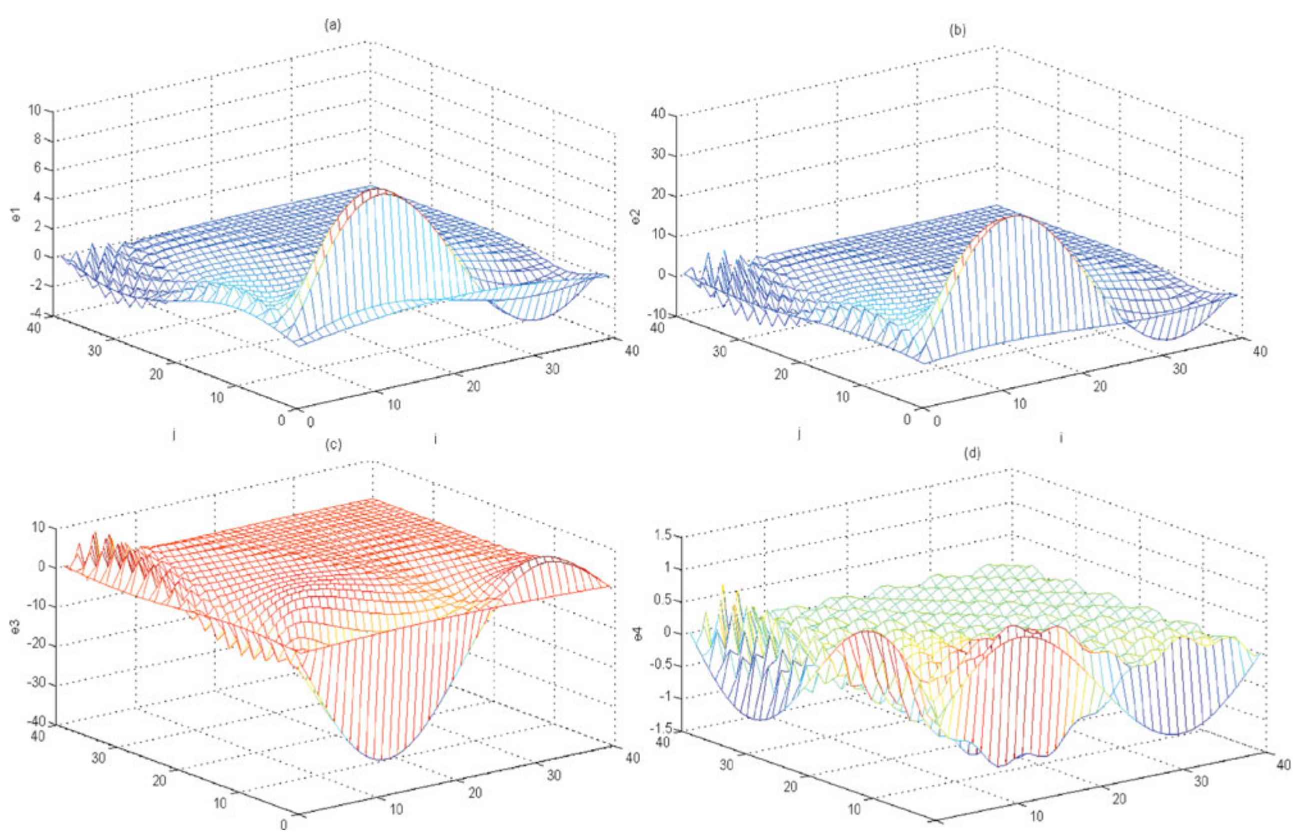

Figure 2. Estimation error $e(i,\ j)$ of the observer in Theorem 3.2 for the 2-D system presented in (48). $e_1(i,\ j)$, $e_2(i.j)$, $e_3(i,\ j)$ and $e_4(i,\ j)$ are shown in (a), (b), (c) and (d), respectively.

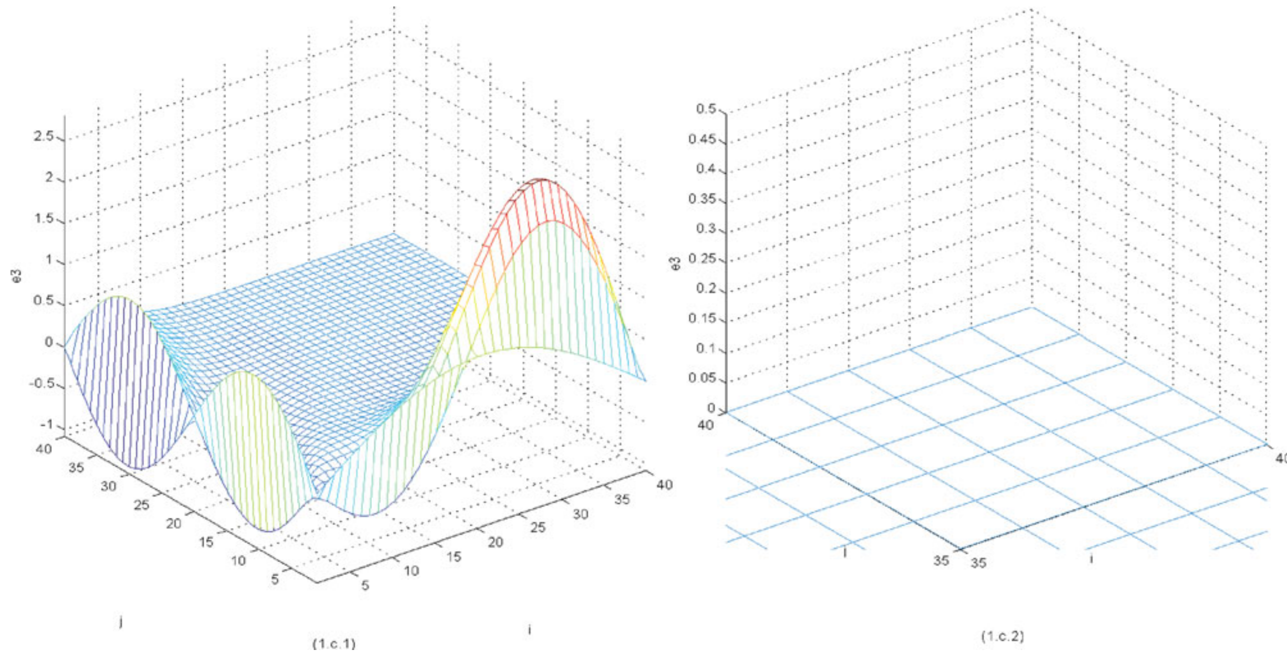

Figure 3. The left subplot shows the fault estimation error in the state equation in Theorem 3.1 and the right subplot shows the zoomed-in error.

to describe the thermal process like heat exchangers, chemical reactors and pipe furnaces (Kaczorek, 1985). Discretise (52) and define

$$\begin{aligned} & T(i,\, j) = T(i\Delta x,\, j\Delta t) \\ & T_x(x,t) = \frac{T(i,j) - T(i-1,j)}{\Delta x} \\ & T_t(x,t) = \frac{T(i,j+1) - T(i,j)}{\Delta t} \\ & x^T(i,\,j) = \left[\, T^T(i-1,\,j)\ \ T^T(i,\,j) \,\right]^T \end{aligned} \tag{53}$$

where Δx and Δt are the space-step and time-step, respectively. Let $\Delta t = 0.1$, $\Delta x = 0.4$, $b = 0$. Substituting (53) into (52), it is easy to yield a 2-D FM-II state space model with

$$\begin{aligned} & A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - \Delta t \end{bmatrix} \\ & M_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 100 \end{bmatrix}^T \end{aligned} \tag{54}$$

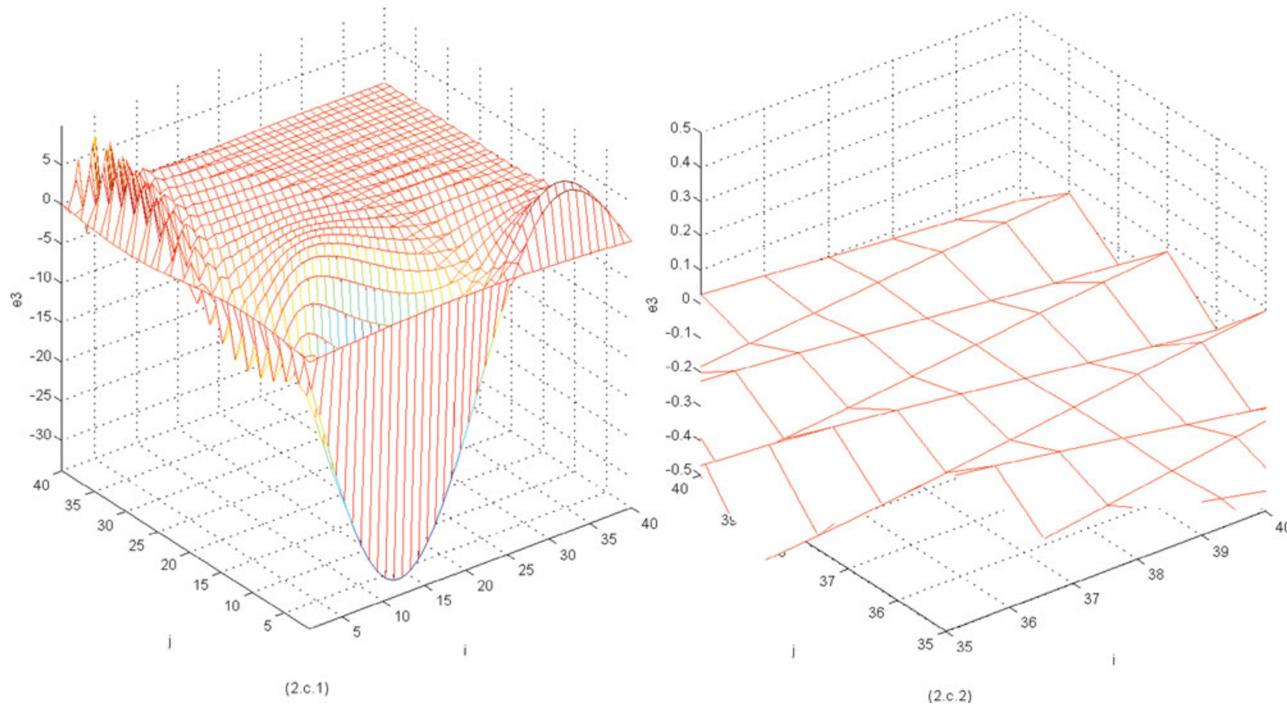

Figure 4. The left subplot shows the fault estimation error in the state equation in Theorem 3.2 and the right subplot shows the zoomed-in error.

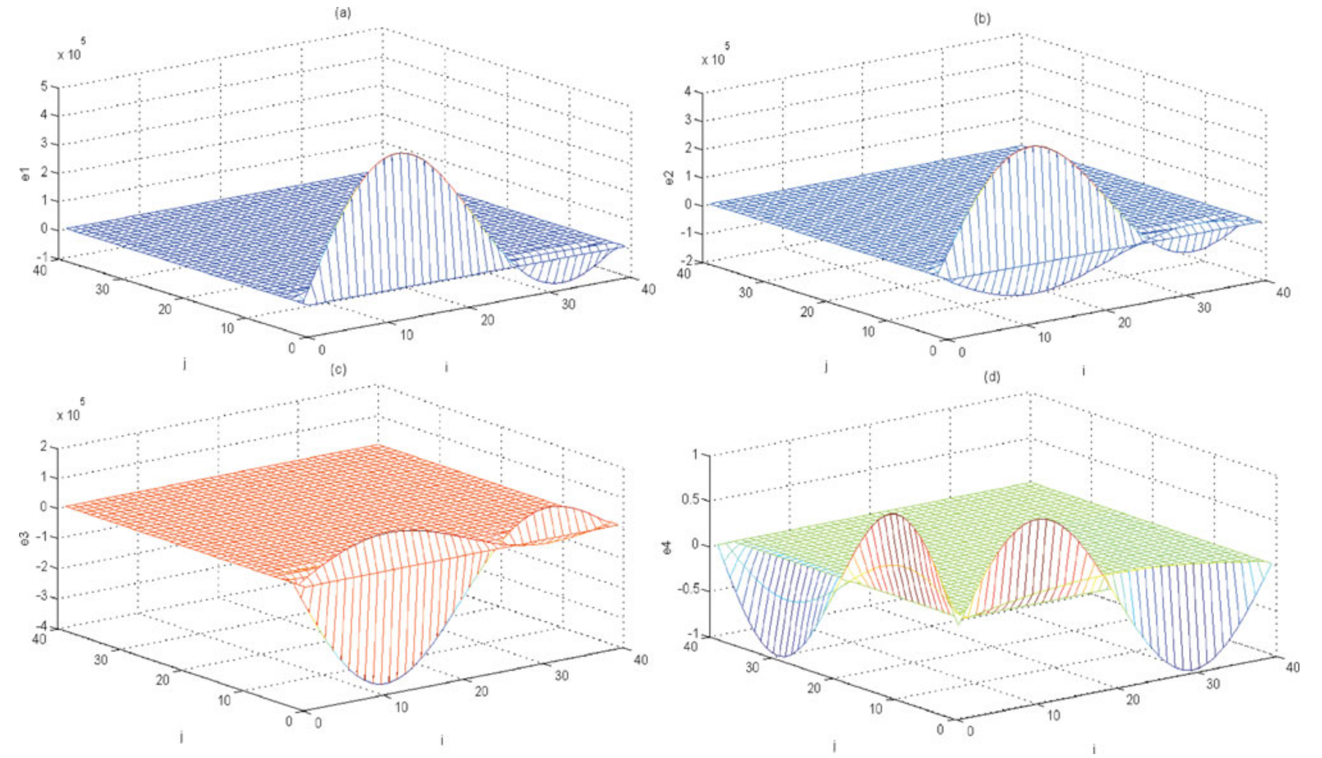

Figure 5. Estimation error $e(i,\ j)$ of the observer in Theorem 3.1 for the 2-D system presented in (54). $e_1(i,\ j)$, $e_2(i.j)$, $e_3(i,\ j)$ and $e_4(i,\ j)$ are shown in subplots (a), (b), (c) and (d), respectively.

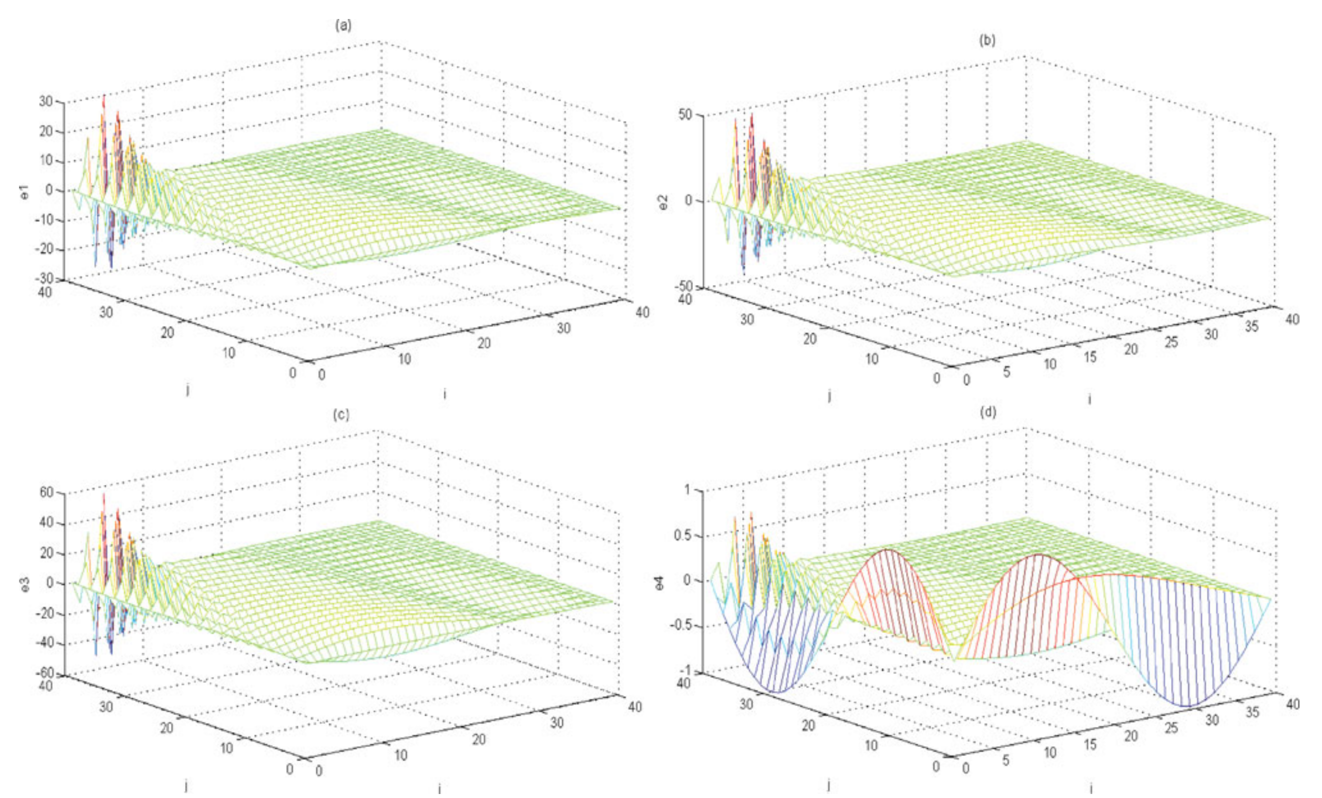

Figure 6. Estimation error $e(i,\ j)$ of the observer in Theorem 3.2 for the 2-D system presented in (54). $e_1(i,\ j)$, $e_2(i.j)$, $e_3(i,\ j)$ and $e_4(i,\ j)$ are shown in subplots (a), (b), (c) and (d), respectively.

and the dynamics of the faults can be assumed as follows:

$$\begin{aligned} f(i,j) &= A_{d1} f(i-1,j) + A_{d2} f\left(i, j-1\right), \\ A_{d1} &= 0.5, A_{d2}{=}0.4 \\ f_s(i,j) &= 20\sin(i+j), h(i,j) = 0.1\sin(i+j) \end{aligned} \quad (55)$$

About the boundary conditions, it is the same case as in Example 2. The obtained estimation error $e(i, j)$ $(0 \le i, j \le 40)$ is shown in Figures 5 and 6. The definitions of $e_1(i, j)$, $e_2(i, j)$, $e_3(i, j)$ and $e_4(i, j)$ are the same as those in Example 1. It is easy to see that the estimation error of both system states and two kinds of faults is asymptotically stable and uniformly ultimately bounded, respectively.

5. Conclusions

The simultaneous estimation of faults in the state equation and measurement equation in 2-D systems are first investigated in this study. For fault in the state equation without disturbance, the sufficient condition for the existence of asymptotically stable observer is presented. For fault in the state equation with disturbance, the sufficient condition for the existence of uniformly ultimately bounded observer is obtained and the upper bound of the estimation error is given. The estimation results can be used for fault reconstruction and active fault-tolerant control. Because there are no related researches about active fault-tolerant control of 2-D linear systems with fault in the state equation and measurement equation, in future, this problem will be discussed. By using the singular systems approach and the Lyapunov stability theory, the observers for 2-D nonlinear systems will also be designed.

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References

Ahn, C. K., Shi, P., & Basin, M. V. (2015). Two-dimensional dissipative control and filtering for Roesser model. *IEEE Transactions on Automatic Control, 60*(7), 1745–1759.

Benzaouia, A., Hmamed, A., & Tadeo, F. (2015). *Two-dimensional systems: From introduction to state of the art*. Cham: Springer.

Bisiacco, M. (1986). On the structure of 2-D observers. *IEEE Transactions on Automatic Control, 31*(7), 676–680.

Bisiacco, M., & Elena Valcher, M. (2004). Unknown input observers for 2D state-space models. *International Journal of Control, 77*(9), 861–876.

Bisiacco, M., & Valcher, M. E. (2006). Observer-based fault detection and isolation for 2D state-space models. *Multidimensional Systems and Signal Processing, 17*(2–3), 219–242.

Boyd, S. P., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*. *15*. Philadelphia: SIAM.

Cao, L., Tao, Y., Wang, Y., Li, J., & Huang, B. (2016). Reliable H∞ control for nonlinear discrete-time systems with multiple intermittent faults in sensors or actuators. *International Journal of Systems Science,* 48 (2), 302–315.

Cheng, J., Fang, M., & Wang, Y. (2016). Subspace identification for closed-loop 2-D separable-in-denominator systems. *Multidimensional Systems and Signal Processing,* doi: 10.1007/s11045-016-0427-y

Ciftcibasi, T., & Yuksel, O. (1983). Sufficient or necessary conditions for modal controllability and observability of Roesser's 2-D system model. *IEEE Transactions on Automatic Control, 28*(4), 527–529.

De Souza, C. E., & Osowsky, J. (2013). Gain-scheduled control of two-dimensional discrete-time linear parameter-varying systems in the Roesser model. *Automatica, 49*(1), 101–110.

Ding, S. X. (2009). Integrated design of feedback controllers and fault detectors. *Annual Reviews in Control, 33*(2), 124–135.

Dong, H., Wang, Z., Ding, S. X., & Gao, H. (2014). Finite-horizon estimation of randomly occurring faults for a class of nonlinear time-varying systems. *Automatica, 50*(12), 3182–3189.

Fornasini, E., & Marchesini, G. (1976). State-space realization theory of two-dimensional filters. *IEEE Transactions on Automatic Control, 21*(4), 484–492.

Fornasini, E., & Marchesini, G. (1978). Doubly-indexed dynamical systems: State-space models and structural properties. *Mathematical Systems Theory, 12*(1), 59–72.

Fornasini, E., & Marchesini, G. (1988). Failure detection in 2D systems. *Control Theory and Advanced Technology, 4*(4), 381–393.

Gao, Z. (2015). Fault estimation and fault-tolerant control for discrete-time dynamic systems. *IEEE Transactions on Industrial Electronics, 62*(6), 3874–3884.

Gao, Z., & Wang, H. (2006). Descriptor observer approaches for multivariable systems with measurement noises and application in fault detection and diagnosis. *Systems & Control Letters, 55*(4), 304–313.

Ghous, I., & Xiang, Z. (2015). Reliable H∞ control of 2-D continuous nonlinear systems with time varying delays. *Journal of the Franklin Institute, 352*(12), 5758–5778.

Givone, D. D., & Roesser, R. P. (1973). Minimization of multidimensional linear iterative circuits. *IEEE Transactions on Computers, 22*(7), 673–678.

Hinamoto, T. (1997). Stability of 2-D discrete systems described by the Fornasini–Marchesini second model. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 44*(3), 254–257.

Kaczorek, T. (1985). *Two-dimensional linear systems*. Berlin: Springer.
Koenig, D. (2005). Unknown input proportional multiple-integral observer design for linear descriptor systems: Application to state and fault estimation. *IEEE Transactions on Automatic Control, 50*(2), 212–217.
Li, L., Ding, S. X., Qiu, J., & Yang, Y. (2017). Real-time fault detection approach for nonlinear systems and its asynchronous T–S fuzzy observer-based implementation. *IEEE Transactions on Cybernetics, 47*(2), 283–294.
Li, L., Ding, S. X., Qiu, J., Yang, Y., & Zhang, Y. (2016). Weighted fuzzy observer-based fault detection approach for discrete-time nonlinear systems via piecewise-fuzzy Lyapunov functions. *IEEE Transactions on Fuzzy Systems, 24*(6), 1320–1333.
Liang, J., Wang, Z., & Liu, X. (2015). H∞ control for 2-D time-delay systems with randomly occurring nonlinearities under sensor saturation and missing measurements. *Journal of the Franklin Institute, 352*(3), 1007–1030.
Lin, Z., & Bruton, L. T. (1989). BIBO stability of inverse 2-D digital filters in the presence of nonessential singularities of the second kind. *IEEE Transactions on Circuits and Systems, 36*(2), 244–254.
Roesser, R. (1975). A discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control, 20*(1), 1–10.
Tao, Y., Shen, D., Wang, Y., & Ye, Y. (2015). Reliable H∞ control for uncertain nonlinear discrete-time systems subject to multiple intermittent faults in sensors and/or actuators. *Journal of the Franklin Institute, 352*(11), 4721–4740.
Wu, L., Yao, X., & Zheng, W. X. (2012). Generalized H2 fault detection for two-dimensional Markovian jump systems. *Automatica, 48*(8), 1741–1750.
Xiao, B., & Yin, S. (2016). Velocity-free fault-tolerant and uncertainty attenuation control for a class of nonlinear systems. *IEEE Transactions on Industrial Electronics, 63*(7), 4400–4411.
Xiao, B., Yin, S., & Kaynak, O. (2016). Tracking control of robotic manipulators with uncertain kinematics and dynamics. *IEEE Transactions on Industrial Electronics, 63*(10), 6439–6449.
Xu, H., Lin, Z., Makur, A., & Xu, L. (2011). Asymptotic unknown input observers for two-dimensional systems . *7th International Workshop on* multidimensional (nD) systems (nDs) (pp. 1–5).
Xu, J., & Yu, L. (2009). Delay-dependent guaranteed cost control for uncertain 2-D discrete systems with state delay in the FM second model. *Journal of the Franklin Institute, 346*(2), 159–174.
Zhao, D., Lin, Z., & Wang, Y. (2015). Integrated state/disturbance observers for two-dimensional linear systems. *IET Control Theory & Applications, 9*(9), 1373–1383.
Zhao, D., Shen, D., & Wang, Y. (2017). Fault diagnosis and compensation for two-dimensional discrete time systems with sensor faults and time-varying delays. *International Journal of Robust and Nonlinear Control.* doi:10.1002/rnc.3742.
Zhao, D., Zhou, D., & Wang, Y. (2015). Sensor fault reconstruction for a class of 2-D nonlinear systems with application to fault compensation. *Multidimensional Systems and Signal Processing, 26*(4), 1061–1080.
Zhong, M., Song, Y., & Ding, S. X. (2015). Parity space-based fault detection for linear discrete time-varying systems with unknown input. *Automatica, 59*, 120–126.